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BODY TENSOR FORMALISM IN FINITE DEFORMATION ELASTICITY AND APPLICATION TO THE TORSION OF A CIRCULAR CYLINDER

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Body tensor formalism in finite deformation elasticity and application to the torsion of a circular cylinder

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BODY TENSOR FORMALISM IN FINITE DEFORMATION ELASTICITY AND APPLICATION TO THE TORSION OF A CIRCULAR CYLINDER

BY

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B.S. NORTHWESTERN UNIVERSITY, 1960

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TO MY MOTHER

TO MY SISTER AND HER FAMILY

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BODY TENSOR FORMALISM IN FINITE DEFORMATION
ELASTICITY AND APPLICATION TO THE TORSION OF
A CIRCULAR CYLINDER

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In this work one develops the body tensor formalism for homogenous, isotropic elastic materials. This formalism is general and not limited to small deformations. One examines the physical laws and the constitutive equation which relates the stress to the strain with the help of the two Lame coefficients and the three third order elastic constants. Although the body tensor formalism has been used before to describe finite deformation elasticity, it has not been used to generalize the constitutive equation for the stress-strain relatioship.

The body tensor formalism and the generalised contitutive equation are applied to the torsion of a right circular cylinder whose length is prevented from changing by the application of an end force.

The solution of the torsion problem leads to a new second order non-linear differential equation which is valid to all orders in the torsion parameter. When linearized this equation can be solved to the first two orders in a dimensionless torsion parameter.

Using the values of the third order elastic constants found in the ultrasonic literature new values are calculated to first order for the radius, the torque and the force at the end of the cylinder for six metallic compounds and three non metallic ones.

CHAPTER I

INTRODUCTION

The description of the theory of linear elasticity is reasonably simple, since one restricts the theory to treat small deformations. This is called the classical theory of elasticity. The deformation of a solid body is described in terms of the strain tensor which can be calculated in either cartesian coordinates or curvilinear coordinates, and the forces are described in terms of the stress tensor, defined with the use of the above coordinates. A linear relationship (generalized Hook's Law) exists between the stresses and strains for a homogenous, isotropic body. This relationship involves the introduction of two constants which describe the properties of the body and are called the Lame coefficients.

The theory of elasticity for large deformations, also called non-linear elasticity,is more complex. Two problems arise: first the description of the deformations of a solid body, second the relationship between the stresses and the strains, which one will call the constitutive equation. The description of the deformations is more difficult since one has to substract from the description of the motion of a point the rigid motion of the body; hence, the description of the deformation in terms of fixed cartesian or curvilinear coordinates is difficult. One way to solve this problem is to express the deformation of a body in terms of a curvilinear coordinate system which moves continiously with the body. See Green and Zerna[8] for example. The advantage of this approach is that tensors which are defined with this coordinate system will automatically describe the deformation of the body. This approach has been followed by A.S. Lodge, who first applied it to elastic liquids[2] and then to solids,[3],[10] A.S. Lodge calls this formalism the body tensor formalism. The same approach has been used by Sedov in a series of works[1],[6]. This approach has also been followed by A.Freed[5] and Oldyrod[29],[30] who calls the coordinates the convected coordinates. One will follow this formalism in this work. Although this formalism is difficult, it is very powerfull and encompasses the description of deformations in terms of Cauchy, Finger, Almansi, and Green strain tensors. See A. S. Lodge[2] for example.

The description of the constitutive equation has traditionaly been done with the introduction of three invariant quantities constructed from the strain tensors and three generalized functions which describe the property of the body and which are functions of these invariant quantities (Green and Zerna[8], Green and Atkins[12], Murnaghan[9]). The problem with this approach is that one needs to know three independent functions to describe the large elastic deformations of a body. Another approach, followed by Murnaghan[9], is to consider the free energy of a deformed body and expand it in terms of the strain tensor invariants. The two Lamé coefficients are associated with the quadratic terms, and three coefficients are associated with the cubic terms and hence, called third order elastic constants, third order Lamé coefficients, or third order elastic moduli. These coefficients were introduced by Landau and Lifshits[7] to describe the anharmonic vibrations of a solid. Even in the linear theory of elasticity the third order terms give rise to measurable effects in the propagation of ultrasonic waves in solids. See the discussion and conclusion section of this work. The advantage of the above approach is that besides the two Lamé coefficients one needs only three more elastic moduli which can be measured by ultrasonic methods. Also if need arises one could extend the theory to the higher order terms and introduce fourth, fifth etc. elastic moduli.

In this work one has applied the body tensor formalism and the constitutive equation to the problem of the simple torsion of a right circular cylinder for which a change in length is prevented. This has been done by M.Reiner[11] with a different constitutive equation, which besides the two Lamé coefficients has a phenomenological term added to it to describe a change in volume and is described by one extra coefficient. Reiner concluded that an axial pressure at the end is required. His result that the change in the radius is a linear function of the radius seems uncertain. Murnaghan[9] has used a matrix formalism to describe the torsion of a right circular cylinder and has used five constants, the two Lame constants and three higher order coefficients. The results of this work for the change in radius are in agreement with his.

Wertheim[24] seems to be the first one to have noticed the non-linearity of the relationship between the torque and the angle of torsion in the torsion of cylinders. Poynting[25],[26] did a series of experiments on the torsion of long wires subjected to loads. He observed a decrease in the radius of the cylinder proportional to the square of the torsion angle and an increase in length of the cylinder also proportional to the square of the torsion angle. (This increase of length is to be added to the increase of length predicted by the linear theory of elasticity). This effect has been called the "Poynting effect" by Trusdell in 1952.

Bell[27] in a review article has summerised the results of Ponyting's experiments and also pointed out that Foux[28] has done the same experiment in 1964. He noted that "in Foux's measurments all the difficulties and fluctuations of data of Poynting's were still present".

This work has been divided into several sections. In Chapter II one describes the body tensor formalism. With the help of a body metric tensor one defines a strain tensor for finite deformations. The stress is also introduced in terms of the body coordinate system.

In Chapter III one examines the physical laws like conversation of mass, change of linear momentum law, conservation of energy, etc.

In Chapter IV one considers the constitutive equation in the body formalism. The approach followed is the introduction of the free energy per unit volume as a function of the strain invariants. By taking the derivatives with respect to strain of this free energy one obtains the stress versus strain relation.

Chapter II, III and IV are general. In Chapter V one has applied the body tensor formalism to the twisting of a right circular cylinder. This chapter is subdivided into three sections. In section A one has considered the general expression of the strain tensor in terms of the body coordinate system and the torsion angle parameter. In section B one has considered the problem of the torsion of a rubber like solid, the material being incompressible. In section C the formalism is applied to the torsion of a cylinder consisting of a material which obeys the constitutive equation derived in Chapter IV. That is to say the constitutive equation is described in terms of the two Lamé coefficients and the three third order elastic constants. End forces are applied to prevent the length of the cylinder from changing. The solution of the problem leads to a non-linear differential equation relating the new radius of the cylinder to the old one. One has linearized this equation and solved it in the first two orders in a small parameter which in this problem is the torsion angle.

In the last chapter, Chapter VI, one has proceeded with some numerical calculations for the new radius, the applied torque and the force at the end of the cylinder. These calculations are of first order in the torsion parameter. To do the numerical calculations the Lamé coefficients and the third order elastic constants for six metallic compounds are used: sintered and resintered tungsten, steel, sintered and resintered molybdenum, magnesium, aluminium, columbium. Three non metallic compounds are also used: pyrex, fused silica and polysterene. These coefficients have been obtained from the literature on ultrasonics.

In Appendix I the relationship between adiabatic and isothermal elastic moduli has been examined. In Appendix II expressions have been developed for the Christoffel symbols and the necessary and sufficient condition for a space to be Euclidian has been examined.

In Appendix III an expression for the divergence of the stress tensor in curvilinear coordinates was obtained.

In Appendix IV one has included the programs E5 and E5' which calculate and plot the new radius, the torque and the force at the end of the cylinder, as functions of the dimensionless torsion angle. The graphs of these quantities and their values are also included.

The content of Chapters II, III and of parts of Chapter IV can be found scattered through the literature[1,2,3,5,6,8,10,12]. The derivation of the constitutive equation from the free energy which includes the third order elastic constants to describe finite deformations of elastic solids is a new result. Using this constitutive equation a more general non-linear differential equation for the torsion of a cylinder which is valid to all order in the torsion parameter is derived. This result is then applied to the calculation of the radius, torque and force at the end of the cylinder, thereby obtaining novel results for several materials.

CHAPTER II

BODY TENSOR FORMALISM

Tensors can be constructed in a body coordinate system as well as in a space coordinate system. Tensors are invariant quantities since they do not depend on the choice of a coordinate system. However their components do depend on the choice of a coordinate system. Such tensors are the stress tensor, the strain tensor and other related tensors.

To illustrate the use of the body coordinate system one describes the elastic deformation of a body in terms of the body coordinate system, and defines a body metric tensor which is then used to construct the strain and stress tensors.

The main distinction between the body tensor formalism and the space formalism is the manifold over which they are defined. The body manifold consits of the material particles, the space manifold consists of an infinite set of spacial places. Body tensor formalisms and space tensor formalism (eg. Oldyrod[29] and Green and Zerna[8]) are different, although as one will see latter the components of the body and space tensors do coincide at a given time.

The position of a point A of a deformable body which can also move in space is usually described by its coordinates x^1, x^2, x^3 in a fixed space coordinate system (S). The coordinates can either be cartesian or curvilinear. The most commonly used curvilinear coordinate systems are the cylindrical and spherical.

The position of A is also described by a parameter t which one will usually assume to be the time.

A body coordinate system is a coordinate system imbedded in the body and it moves and deforms as the body moves in space and deforms. The coordinates of the point A are S^1, S^2, S^3 and are fixed, that is to say do not change as functions of time t. The coordinate system S^1, S^2, $S^3 \leftrightarrow (B)$ is usually a curvilinear coordinate system but any one to one correspondence between points in a body and real numbers could also be considered. An illustration of the above is given in Fig. (1). Consider a point B in the neighborhood of A where A has coordinates (S^1,S^2,S^3). The body coordinates of point B are $S^i + dS^i$ (ie. $S^1 + dS^1$, $S^2 + dS^2$, $S^3 + dS^3$). One can define in the body coordinate system a covariant basis formed by three vectors $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ defined by:

$$\vec{e}_1 = \frac{\partial \vec{r}}{\partial S^1} \quad , \quad \vec{e}_2 = \frac{\partial \vec{r}}{\partial S^2} \quad , \quad \vec{e}_3 = \frac{\partial \vec{r}}{\partial S^3} \tag{II-1}$$

where $\vec{r}$ is the position vector of point A in the fixed space coordinate system. From the definition above we see that $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ are vectors tangent to the body coordinate lines S^1, S^2, S^3. This is illustrated in fig (1). Let the basis vector be $\vec{e}_{1,o}$ $\vec{e}_{2,o}$ $\vec{e}_{3,o}$ at time t_o and $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ at time t. From the description above one gets, if $d\vec{r}_o$ is the vector between A and B at time t_o and dr the vector between A and B at time t,

$$d\vec{r}_o = dS^1\vec{e}_{1,o} + dS^2\vec{e}_{2,o} + dS^3\vec{e}_{3,o} \tag{II-2}$$

and

$$d\vec{r} = dS^1\,\vec{e}_1 + dS^2\,\vec{e}_2 + dS^3\,\vec{e}_3\,. \qquad \text{(II-3)}$$

For simplicity one can use the repeated index summation convention in all the following equations, that is to say sum over repeated latin indexes over the values 1, 2 and 3. One can use a "$\hat{i}$" if one does not want a sum over a repeated index "i". Hence, equations (II-2) and (II-3) can be rewritten as

$$d\vec{r}_o = dS^i\,\vec{e}_{i,o}\;, \quad d\vec{r} = dS^i\,\vec{e}_i\,. \qquad \text{(II-3')}$$

The equations (II-2) and (II-3) have a geometrical interpretation. The vector $\vec{AB}$ of fig (1) is the vectorial sum of three vectors directed along the basis vectors $\vec{e}_{1,o}$, $\vec{e}_{2,o}$, $\vec{e}_{3,o}$ or $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$. Usually the basis vectors are not orthogonal, and hence, the vector $\vec{AB}$ will be the diagonal of a parallepiped of sides parallel to the basis vectors. This added complication is a slight drawback of the use of the body coordinate systems. By taking the scalar product of $d\vec{r}_o$ with itself and $d\vec{r}$ with itself one can calculate the square of the length of AB at time to and at time t.

$$dr_o^2 = \gamma_{ij,o}\, dS^i dS^j \qquad \text{(II-4)}$$

and

$$dr^2 = \gamma_{ij}\, dS^i dS^j \qquad \text{(II-5)}$$

where

$$\gamma_{ij,o} = \vec{e}_{i,o}\cdot\vec{e}_{j,o} \tag{II-6}$$

and

$$\gamma_{ij} = \vec{e}_i\cdot\vec{e}_j \quad . \tag{II-7}$$

The γ's defined in (II-6) and (II-7) and (II-4) and (II-5) will be called body tensor metrics and are the important quantities in dealing with the description of deformations. From (II-4), (II-5), since dr^2 usually is a function of time and dS^i are time independent, one concludes that the γ_{ij} are time dependent. Hence, the notation $\gamma_{ijo} = (\gamma_{ij})_{t=to}$ and $\gamma_{ij} = (\gamma_{ij})_t$. Also from (II-4),(II-5) or (II-6), (II-7) one concludes that the γ's are symmetric in the indexes i and j,

$$\gamma_{ij} = \gamma_{ji}. \tag{II-8}$$

Further, in future equations one will use γ_0 and γ to represent the matrices which have elements γ_{ijo} and γ_{ij}.

Also equations (II-6) and (II-7) give a geometrical interpretation of the body tensor metric, its elements being the scalar products of the basis vectors. Furthermore, the volume of the parallelipiped formed by the projection of $\vec{dr}$ along $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$ is given by

$$V = \vec{e}_1 \cdot (\vec{e}_2 \times \vec{e}_3)\, dS^1\, dS^2\, dS^3. \tag{II-9}$$

From (II-7) one gets

$$(\vec{e}_1 \cdot (\vec{e}_2 \times \vec{e}_3))^2 = \det \gamma. \tag{II-10}$$

At this point one should mention the tensor nature of the previous formulas. The dS^i are the contravariant components of the vector $\vec{dr}$ (contravariant quantities have upper indexes, covariant lower ones). γ_{ij} are the covariant components of a second rank tensor, the body metric tensor. One can also start with a contravariant body metric tensor. For this purpose one defines contravariant basis vectors $\vec{e}^1$, $\vec{e}^2$, $\vec{e}^3$ as reciprocals of the covariant basis vectors $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ by

$$\vec{e}^1 = \frac{\vec{e}_2 \times \vec{e}_3}{\vec{e}_1 \cdot \vec{e}_2 \times \vec{e}_3} \tag{II-11}$$

with $\vec{e}^2$ and $\vec{e}^3$ obtained by cyclic permutation from (II-11). The contravariant basis vectors from the definition are orthogonal to the covariant basis vectors.

From the definition (II-11) one sees that the covariant basis and contravariant ones are orthonormal, ie.

$$\vec{e}_i \cdot \vec{e}^j = \delta_i^j \tag{II-12}$$

where δ_i^j is the Kronecker delta function defined by

$$\begin{aligned} \delta_i^j &= 0 \qquad i \neq j \\ \delta_i^j &= 1 \qquad i = j \quad . \end{aligned} \tag{II-13}$$

One can also define a contravariant body metric tensor which has elements

$$\gamma^{ij} = \vec{e}^i \cdot \vec{e}^j \quad . \tag{II-14}$$

One can easily show that the matrix which has elements γ^{ij} is the inverse of the matrix γ_{ij}, hence, the notation

$$(\gamma^{-1})_{ij} = \gamma^{ij}. \tag{II-15}$$

In the formalism which is to follow one will use covariant and contravariant body tensors. One should note that dS^i, the covariant components of the infinitesimal vector $d\vec{r}$ are not exact differentials (Sedov [1]).

The kinematics of a body is described by the motion of points of the body. By motion one means possible motion of the center of mass of the body, rotation of the body, and deformations. Since the coordinates S^1, S^2, S^3 of a point A of the body are fixed in the body coordinate system (B), the motion of the point is described by the coordinates x^1, x^2, x^3 in the space coordinate system (S)

$$x^i = x^i(S^1, S^2, S^3, t). \tag{II-16}$$

Given the body coordinates S^1, S^2, S^3 of a point and the parameter t (usually time) one can calculate x^1, x^2, x^3. The correspondence between S^i and x^i must be one to one; hence, the inverse function exists,

$$S^i = S^i\ (x^1, x^2, x^3, t). \tag{II-17}$$

In the space coordinate system (S) the square of the length of $\overrightarrow{AB}$ is given by

$$dr^2 = g_{ij}(x^1, x^2, x^3)\ dx^i\ dx^j \tag{II-18}$$

where g_{ij} are the covariant components of the space metric tensor. With the help of the transformation equation (II-16) the above equation can be transformed to

$$dr^2 = g_{ij}(x^i(S^1, S^2, S^3, t))\left(\frac{\partial x^i}{\partial S^k}\right)_t \left(\frac{\partial x^j}{\partial S^m}\right)_t dS^k dS^m. \tag{II-19}$$

Since the equation above is general, one immediately obtains

$$dr_0^2 = g_{ij}(x^{\hat{i}}(s^1,s^2,s^3,t_0))\left(\frac{\partial x^i}{\partial s^k}\right)_{t_0}\left(\frac{\partial x^j}{\partial s^m}\right)_{t_0} ds^k ds^m. \quad \text{(II-20)}$$

One now looks for a way to relate the description of the system in a body coordinate system (B) and in a space coordinate system (S). The body coordinate system (B) is arbitrary. However, to make the formalism more tractable it is useful to choose a body coordinate system which coincides with a space coordinate system at either time t_0 or t. One can say that one has an isomorphism between the two coordinate systems in either state t or state t_0.

Consider the isomorphism in state t. The body coordinate system (B) and the space coordinate system (S) coincide at time t; hence, at time t $x^i = S^i$ (fig.(2)). Therefore $\left(\frac{\partial x^i}{\partial s^k}\right)_t = \delta^i_k$ and if one compares (II-19) and (II-5) one obtains

$$\gamma_{ij}(S^{\hat{i}}, t) = g_{ij}(x^{\hat{i}}). \quad \text{(II-21)}$$

The body metric tensor and the space metric tensor are equal. Furthermore, one can also express $g_{ij}(x^{\hat{i}})$ in another coordinate system, $\bar{x}^i$, for example. This follows the usual tensor transformation laws or

$$g_{ij}(x^{\hat{i}}) = \frac{\partial \bar{x}^m}{\partial x^i}\frac{\partial \bar{x}^n}{\partial x^j}\, \bar{g}_{mn}(\bar{x}^{\hat{i}}) \quad \text{(II-22)}$$

where $\bar{g}_{mn}(\bar{x}^i)$ is the metric tensor in a space coordinate (S′) of coordinates $\bar{x}^i$.Now comparing (II-4) and (II-20) one obtains

$$\gamma_{ij}(s^{\hat{i}}, t_0) = g_{mn}(x^{\hat{i}}(s^{\hat{i}},t_0))\left(\frac{\partial x^m}{\partial s^i}\right)_{t_0}\left(\frac{\partial x^n}{\partial s^j}\right)_{t_0}. \quad \text{(II-23)}$$

Introduce a new independent variable r^i which is the value of x^i at time t_o (Lagrangian variable). Hence, from (II-16) one obtains,

$$r^i = r^i(S^1, S^2, S^3, t_o) \qquad \text{(II-24)}$$

and from (II-23)

$$\gamma_{ij}(S^{\hat{i}}, t_o) = g_{mn}(r^{\hat{i}}(S^{\hat{i}}, t_o)) \left(\frac{\partial r^m}{\partial S^i}\right)_{t_o} \left(\frac{\partial r^n}{\partial S^j}\right)_{t_o} \qquad \text{(II-25)}$$

Consider an isomorphism in state t_o.

The body coordinate system (B) and the space coordinate system (S) coincide at time t_o, Fig (3). Hence, $x^i(t_o) = S^i$ and since $x^i(t_o) = r^i$, one obtains $S^i = r^i$. By comparing equations (II-4) and (II-20) one obtains

$$\gamma_{ij,o} = g_{ij}(r^{\hat{i}}) . \qquad \text{(II-26)}$$

If one compares equations (II-5) and (II-19), one gets

$$\gamma_{ij}(S^{\hat{i}}, t) = g_{mn}(x^{\hat{i}}(S^{\hat{i}}, t)) \left(\frac{\partial x^m}{\partial S^i}\right)_t \left(\frac{\partial x^n}{\partial S^j}\right)_t \quad . \qquad \text{(II-27)}$$

One should make a note of the difference between the space metric tensor and the body metric tensor. The space metric tensor is time independant and the body metric tensor depends on time.

The deformation of a body is determined by the quantity $\gamma - \gamma_0$ since for a rigid body motion $dr^2 = dr_0^2$ and $\gamma - \gamma_0 = 0$. Hence, $\gamma - \gamma_0$ as expressed in the body formalism will describe the deformations of a body. It can be calculated from geometrical considerations from (II-6) and (II-7) or by calculations from (II-21), (II-23) if one uses an isomorphism in state t or from (II-26) and (II-27) if one uses an isomorphism in state t_0. The body tensor formalism being general (Lodge [2,3]) one does not need to determine before hand which isomorphism is to be used.

In this work one starts with equation (II-16) describing the motion of particles of the body and then calculates γ and γ_0. In other types of problems one can start with given stresses and then calculate the strains and then γ. One must be sure that γ represents a metric tensor for a Euclidian space. This compatibility condition is equivalent to the vanishing of the Riemann - Christoffel tensor (Lodge [3], Freed[5] and Appendix II).

To do this one has to introduce some quantities from tensor analysis. The Christoffel symbols of the first kind are three index quantities (not a tensors) defined by

$$\Gamma_{i,jk} = \frac{1}{2}\left(\frac{\partial \gamma_{ij}}{\partial s^k} + \frac{\partial \gamma_{ik}}{\partial s^j} - \frac{\partial \gamma_{jk}}{\partial s^i}\right) \quad . \qquad \text{(II-28)}$$

The Christoffel symbol of the second kind is defined by

$$\Gamma^i_{jk} = \gamma^{ir}\,\Gamma_{r,jk} \qquad \text{(II-29)}$$

because of the symmetry of the γ's Γ_{ijk} and Γ^{i}_{jk} are symmetric in j and k. The vanishing of the Riemann-Christoffel tensor is equivalent to

$$\frac{\partial \Gamma_{i,jl}}{\partial S^k} + \Gamma^{r}_{jk} \Gamma_{r,il} = \frac{\partial \Gamma_{i,jk}}{\partial S^l} + \Gamma^{r}_{jl} \Gamma_{r,ik} \quad . \tag{II-30}$$

The description of the deformations in the body coordinate formalism because it involves isomorphism at time t or time t_0 is somehow different from the formalism of the deformation in classical elasticity. However, the description of stress is the same in a body formalism or in classical elasticity. However, one has to keep in mind that one has a curvilinear coordinate system and care has to be taken to calculate quantities like gradients and divergence of vector and tensor fields.

The traction on a surface (force per unit area considered positive for a normal going from medium II-to medium I) is $\vec{f}$. This is the force exerted by medium I on medium II. If $\vec{\nu}$ is a unit normal to the surface Σ as shown in fig(4) then one defines a stress tensor by its contravariant components π^{ij} with

$$f^i = \pi^{ij} \nu_j . \tag{II-31}$$

A surface Σ can be specified by an equation like $\phi(S^1,S^2,S^3)=$ constant. The normal to the surface is given by

$$\vec{\nu} = \frac{\nabla\phi}{(\nabla\phi\cdot\gamma^{-1}\cdot\nabla\phi)^{\frac{1}{2}}} \quad , \tag{II-32}$$

and the covariant components of the gradient are given by

$$\nabla\phi_i = \frac{\partial\phi}{\partial S^i} \quad . \tag{II-33}$$

The denominator in (II-32) insures that the vector $\vec{\nu}$ is a unit vector $(\nu^i\ \nu_i = 1)$.

In the absence of internal angular momenta and zero surface torques the conservation of angular momentum holds, and one can show that the stress tensor must be symmetric (Sedov[1])

$$\pi^{ij} = \pi^{ji} \ . \tag{II-34}$$

From the definition (II-31) it is clear that the stress tensor components are functions of the body coordinate S^1, S^2, S^3 and the time t.

Since one has introduced the concepts of deformations $(\gamma - \gamma_0)$ and of stress (π), the physical laws that these quantities must obey can now be examined.

CHAPTER III

PHYSICAL LAWS

The well known physical laws of conservation of mass, change of linear momentum, change of angular momentum, conservation of energy and the concept of entropy are usually written down in terms of a space coordinate system and can be found in most books of continium mechanics (Prager[4], for example). The formulations of these laws in a body coordinate system will now be examined. One also has to keep in mind that since these laws are invariant under a change of coordinate system, they must be written in terms of tensor quantities which will then automatically insure the invariance of the laws. At this point one will not derive all these laws but give heuristic arguments for their validity. Derivations of these laws can be found in Lodge[2], [3], Freed[5] and Sedov.[1],[6]

First consider the conservation of mass law. The infinitesimal volume of a body between the surfaces S^1=constant, S^1+dS^1= constant, S^2=constant, S^2+dS^2=constant, S^3=constant and S^3+dS^3=constant will be considered. The mass of this element Δm is invariant with respect to time. Of course, it can depend on S^1, S^2, S^3. The density ρ (S^i, t) in a body coordinate system is defined the usual way as

$$\rho\,(S^i,t) = \frac{\Delta m}{\Delta V} \qquad \text{(III-1)}$$

Since V is given by (II-9) and (II-10),

$$\rho \sqrt{\det \gamma} \;\; dS^1 \, dS^2 \, dS^3 = \Delta m \; ; \tag{III-2}$$

hence, from (III-2) one concludes that $\rho \sqrt{\det \gamma}$ is a constant. By taking the time derivative of the above equation one gets

$$\frac{\partial \rho}{\partial t} \sqrt{\det \gamma} + \frac{1}{2} \rho \frac{1}{\sqrt{\det \gamma}} \frac{\partial \det \gamma}{\partial \gamma_{ij}} \frac{\partial \gamma_{ij}}{\partial t} = 0 \; . \tag{III-3}$$

Making use of a result by Lodge [(2)] which can easily be shown to be true by expanding the determinant one obtains

$$\frac{\partial \det \gamma}{\partial \gamma_{ij}} = (\det \gamma) \, \gamma^{ji}$$

Since the γ's are symmetric,

$$\frac{\partial \rho}{\partial t} + \frac{1}{2} \rho \; \mathrm{Trace} \left(\gamma^{-1} \frac{\partial \gamma}{\partial t} \right) = 0 \tag{III-4}$$

Equation (III-4) gives the time rate of change of the density as expressed in a body coordinate system. An obvious result being that for a rigid body motion, since γ is constant, the density ρ is also constant.

Now consider the linear motion of the above infinitesimal object of mass Δm. One can use Newton's law and write

$$\rho \, a^i = F^i + \nabla_j \, \pi^{ij} \tag{III-5}$$

where $\nabla_j \pi^{ij}$ is the divergence (in the body coordinate system) of the stress tensor, and F^i is the contravariant component of the body force per unit volume. The above equation resembles the equation of motion in the space coordinate system, but one has to calculate the divergence of the stress tensor in a curvilinear coordinate system and calculate the acceleration components a^i in the reference system of the element at time t.

To clarify the last statement let's look at the motion of a point A of the body in a time dt, and for simplicity look at a two dimensional drawing (Fig (5)). In this time interval the point moves from A(t) to A(t + dt), and the vector $\vec{dr} = \overrightarrow{A(t)\ A(t+dt)}$ has components dx^i in the $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ basis system. For simplicity one can choose the space system x^i and body systems S^i to be isomorphic at time t. Hence, the velocity is given by

$$\vec{V} = \frac{\partial x^i}{\partial t}(S^i,t)\vec{e}_i = V^i(S^i,t)\vec{e}_i \quad , \qquad \text{(III-6)}$$

and the acceleration is obtained by taking the derivative with respect to time at constant S^i of the above equation

$$\vec{a} = \frac{\partial V^i}{\partial t}(S^i,t)\vec{e}_i + V^i(S^i,t)\frac{d\vec{e}_i}{dt} \quad . \qquad \text{(III-7)}$$

The change of basis vectors is given in terms of Christoffel symbols of the second kind (see Sedov[1] and Appendix II) by

$$\frac{d\vec{e}_i}{dt} = \frac{\partial \vec{e}_i}{\partial x^j}\frac{\partial x^j}{\partial t} = V^j\,\Gamma^{\alpha}_{ij}\,\vec{e}_\alpha \quad . \qquad \text{(III-8)}$$

Substituting (III-8) into (III-7) and the result into (III-5) one gets

$$\rho\left(\frac{\partial v^i}{\partial t} + v^m v^n \Gamma^i_{mn}\right) = F^i + \nabla_j \pi^{ij} \ . \tag{III-9}$$

Keeping in mind that

$$v^i = \frac{\partial x^{\hat{i}}}{\partial t}\left(s^{\hat{i}}, t\right) \tag{III-9'}$$

the divergence of π can be written as (Lodge[7] and Appendix III).

$$\nabla_j \pi^{ij} = \frac{\partial \pi^{ij}}{\partial s^j} + \pi^{jk}\Gamma^i_{jk} + \pi^{ij}\Gamma^k_{jk} \ . \tag{III-10}$$

Equation (III-10) can be rewritten as

$$\nabla_j \pi^{ij} = \frac{1}{\sqrt{\det \gamma}}\frac{\partial}{\partial s^j}\left(\sqrt{\det \gamma}\, \pi^{ij}\right) + \pi^{jk}\Gamma^i_{jk} \ . \tag{III-11}$$

The equation of motion (III-9) which can be evaluated with the help of equations (III-9') and (III-10) is slightly more complex than the equation of motion in an orthogonal space coordinate system. However, in the case of elastostatics all the velocity terms drop out. Furthermore, if one has no outside body forces, the resulting static equation is

$$\nabla_j \pi^{ij} = 0 \tag{III-12}$$

which is just the generalization to curvilinear coordinate systems of the orthogonal space coordinate system equation

$$\frac{\partial p^{ij}}{\partial x^{j}} = 0 \quad . \tag{III-13}$$

The change of angular momentum equation is rather complex in a body coordinate system. (Sedov [(1), (6)]) It reduces in the absence of internal torques, to the simple condition of symmetry of the stress tensor, π^{ij} . This result is to be expected since at time t π^{ij} and p^{ij} are isomorphic, and one knows that p^{ij} is a symmetric tensor.

Let's consider the conservation of energy principle including the temperature as a state variable to make the analysis more general. The state of a body at time t can be specified in terms of the deformations which are functions of γ_0 and γ and also of temperature T. One will also consider the thermodynamic quantities per unit volume contrary to some other authors (Lodge[(2)]) who consider these quantities per unit mass. Hence,

$$U = U(\gamma, \gamma_0, T, S^i) \qquad S = S(\gamma, \gamma_0, T, S^i) \quad \tilde{F} = \tilde{F}(\gamma, \gamma_0, T, S^i) \tag{III-14}$$

where U is the internal energy per unit volume, S the entropy and $\tilde{F}$ the free energy.

Another approach (A. Freed[(5)]) is to consider the Gibbs free energy ϕ which is a function of the stress variables instead of the strains.

$$\phi = \phi\ (\ \gamma_0,\ \pi,\ T,\ S^i\ ). \tag{III-15}$$

One can easily relate the two formalisms.

The work done, on an infinitesimal volume consists of two parts. One part gives the change of the kinetic energy of the volume, and the other part gives the change of internal energy of the volume. Only the latter will be considered. One assumes that there are no electromagnetic forces in our system. If this is not the case, one can easily add electromagnetic terms to the work done. For a fluid it is -pdv. For a deformable body it is

$$dW = \frac{1}{2}\, \pi^{ij}\, d\gamma_{ji} \; = \; \frac{1}{2} \, \mathrm{Trace}\,(\, \pi \, d\gamma \,) \; . \tag{III-16}$$

The above expression derived by Lodge[2], [3] and Freed [5] is isomorphic to the well known expression of classical elasticity $dW = p^{ij}\,\varepsilon_{ji}$ where ε_{ij} is the infinitesimal strain tensor.

One will consider reversible processes. Processes which are irreversible like internal friction can still be considered if changes of entropy are calculated for the equivalent reversible processes. One has, therefore, the well known thermodynamical relations

$$dU = T\, dS + dW \; . \tag{III-17}$$

Since

$$\widetilde{F} = U - T\,S \; , \tag{III-18}$$

$$d\tilde{F} = - S\ dT + dW\ . \qquad \text{(III-19)}$$

From (III-17) and (III-19) with the help of (III-16) one obtains

$$\Pi^{ij} = 2\left(\frac{\partial \tilde{F}}{\partial \gamma_{ij}}\right)_T \qquad \text{(III-20)}$$

and

$$\Pi^{ij} = 2\left(\frac{\partial U}{\partial \gamma_{ij}}\right)_S\ . \qquad \text{(III-21)}$$

Equations (III-20) and (III-21) are the basic equations for the constitutive equations of a deformable body. For adiabatic processes (like elastic wave propagation) one should use equation (III-21). Of course one can easily go from a formalism using the free energy to one using the internal energy with the help of relation (III-18). It should however be pointed out that the elastic parameters (Young's modulus and Poisson ratio for example) will be different for isothermal or adiabatic processes (Landau[7]).

If one considers processes with small changes of temperature, the free energy can be expanded in terms of the temperature difference $T-T_0$ where T_0 is the temperature in the strain free state. Hence,

$$\tilde{F}(\gamma,\gamma_0,S^i,T) = F_0(T)+C(\gamma,\gamma_0)(T-T_0)+F(\gamma,\gamma_0). \qquad \text{(III-22)}$$

In the above equation $\tilde{F}$ is the free energy and $F_0(T)$ the free energy of the body in the strain free state.

The above quantities are also functions of S^i, but for simplicity one will assume homogenous systems and drop the S^i dependence. If need be, one can generalize the theory to inhomogenous systems by simply restoring this dependence. $C(\gamma, \gamma_0)$ is a scalar function of the strain and $F(\gamma, \gamma_0)$ the free energy of the strained body at temperature T_0.

Furthermore, one can simplify equation (III-22) by considering C to be a linear function of the strain. The covariant strain is $\gamma_{ij} - \gamma_{ij,o}$. γ_{ij} is also a measure of the deformation of the body, but one should consider the strain function which reduces to zero in the undeformed state. Hence, the choice of $\gamma_{ij} - \gamma_{ij,o}$.

Let us define a mixed strain tensor ϵ^i_j by

$$\epsilon^i_j = \gamma_0^{ik} (\gamma_{kj} - \gamma_{kj,o}) . \tag{III-23}$$

The only first order scalar function of the strain is, therefore,

$$\epsilon^i_i = \gamma_0^{ik}(\gamma_{ki} - \gamma_{ki,o}) = \text{Trace } \gamma_0^{-1}(\gamma - \gamma_0). \tag{III-24}$$

Therefore one obtains from (III-22)

$$F(\gamma,\gamma_0,T) = F(T_0) - K\alpha(\text{Trace}\gamma_0^{-1}(\gamma-\gamma_0))(T-T_0) + F(\gamma,\gamma_0) \tag{III-25}$$

where in (III-25) the coefficient of the trace is written in terms of the constants K and α . K is the bulk modulus written in terms of the Lamé coefficients μ and λ

$$K = \lambda + \frac{2}{3}\mu \ ,$$

and α is the coefficient of thermal expansion. The - sign corresponds to the usual expansion of materials under increase of temperature.

Substituting (III-25) into (III-20) one obtains

$$\Pi^{ij} = -2K\alpha(T-T_0)\gamma_0^{ij} + 2\frac{\partial F}{\partial \gamma_{ji}} \ . \tag{III-26}$$

One will further restrict processes to be quasi-equilibrium processes in terms of temperatures. That is to say, the rate of change of stresses will be small enough so that the system can be considered to undergo an isothermal process. Therefore, the first term of the right hand side of equation (III-26) can be neglected and one obtains

$$\Pi^{ij} = 2\frac{\partial F}{\partial \gamma_{ji}} \ . \tag{III-27}$$

Therefore, in what is to follow one needs only to consider the strain part of the free energy of the system since one will be dealing with isothermal processes.

CHAPTER IV

CONSTITUTIVE EQUATIONS

Let's consider an isotropic, homogenous system (see remark following equation (III-22) for inhomogenous systems). One will also consider isothermal processes; hence, as discussed in the previous chapter one needs only to consider the free energy of the strained body.

The strain free energy is a scalar function of the mixed strain tensor defined in the previous chapter, equation (III-23) or

$$\varepsilon_j^i = \gamma_o^{ik} \ (\gamma_{kj} - \gamma_{kj,o}).$$

Three independant scalars functions can be constucted from the above tensor. The two sets commonly chosen (Sedov [1]) are the set J_1, J_2, J_3 and the set J_1^*, J_2^*, J_3^* with

$$\begin{aligned} J_1 &= \varepsilon_i^i \\ J_2 &= \varepsilon_j^i \ \varepsilon_i^j \qquad\qquad \text{(IV-1)} \\ J_3 &= \varepsilon_k^i \ \varepsilon_l^k \ \varepsilon_i^l \end{aligned}$$

and

$$
\begin{aligned}
J_1^* &= \varepsilon_i^i \\
J_2^* &= \tfrac{1}{2}(\varepsilon_i^i \varepsilon_j^j - \varepsilon_j^i \varepsilon_i^j) \\
J_3^* &= \mathrm{Det}(\varepsilon_j^i) \ .
\end{aligned}
\tag{IV-2}
$$

With the relationship between the two sets given by.

$$
\begin{aligned}
J_1 &= J_1^* \\
J_2 &= J_1^{*2} - 2\, J_2^* \\
J_3 &= 3\, J_3^* + J_1^{*3} - 3\, J_1^*\, J_2^* \ .
\end{aligned}
\tag{IV-3}
$$

One can start with the set J_1, J_2, J_3; hence, $F{=}F\ (J_1, J_2, J_3)$. From the equation relating the stress to the derivative of the strain free energy at constant temperature, equation (III-27) or

$$\Pi^{ij} = 2\,\frac{\partial F}{\partial \gamma_{ji}} \tag{IV-3'}$$

one obtains

$$\Pi^{ij} = 2\,\frac{\partial F}{\partial J_1}\frac{\partial J_1}{\partial \gamma_{ji}} + 2\,\frac{\partial F}{\partial J_2}\frac{\partial J_2}{\partial \gamma_{ji}} + 2\,\frac{\partial F}{\partial J_3}\frac{\partial J_3}{\partial \gamma_{ji}} \ . \tag{IV-4}$$

With the help of the chain rule for derivatives

$$\frac{\partial}{\partial \gamma_{ji}} = \frac{\partial}{\partial \epsilon_m^k}\,\frac{\partial \epsilon_m^k}{\partial \gamma_{ji}} \tag{IV-5}$$

and using equation (III-23) one obtains

$$\frac{\partial \epsilon^k_m}{\partial \gamma_{ij}} = \gamma_0^{ki} \delta^j_m \quad . \tag{IV-6}$$

Therefore,

$$\pi^{ij} = 2\frac{\partial F}{\partial J_1} \gamma_0^{ji} + 4\frac{\partial F}{\partial J_2} \epsilon^j_k \gamma_0^{ki} + 6\frac{\partial F}{\partial J_3} \epsilon^j_m \epsilon^m_k \gamma_0^{ki} \tag{IV-7}$$

or with the definition of equation (III-23) one obtains the following expression for the stress tensor

$$\pi^{ij} = 2\frac{\partial F}{\partial J_1} \gamma_0^{ji} + 4\frac{\partial F}{\partial J_2} \gamma_0^{im} (\gamma_{mk} - \gamma_{mko}) \gamma_0^{kj}$$
$$+ 6\frac{\partial F}{\partial J_3} \gamma_0^{is} (\gamma_{sm} - \gamma_{smo}) \gamma_0^{mp} (\gamma_{pk} - \gamma_{pko}) \gamma_0^{kj}. \tag{IV-8}$$

One notices from equation (IV-8) that as expected $\pi^{ji} = \pi^{ij}$. The above relation can be written in terms of matrices in a simpler form as

$$\pi = 2\frac{\partial F}{\partial J_1} \gamma_0^{-1} + 4\frac{\partial F}{\partial J_2} \gamma_0^{-1}(\gamma - \gamma_0)\gamma_0^{-1} + 6\frac{\partial F}{\partial J_3} \gamma_0^{-1}(\gamma - \gamma_0)\gamma_0^{-1}(\gamma - \gamma_0)\gamma_0^{-1}. \tag{IV-9}$$

Equation (IV-9) is a general equation giving the stress tensor in matrix form. One can also use the long form equation (IV-8). It involves a linear term in $\gamma_0^{-1}(\gamma-\gamma_0)$ and also a square term in the same variable. The fact that higher order terms do not appear is not surprising since from the Hamilton-Cayley Theorem (see Prager[4] for example) cubic and higher order terms can be expressed in terms of the lower order terms.

Another approach, similar to the above can be found in Lodge[2] and Green and Zerna[8]. The basic invariants being

$$
\begin{aligned}
I_1 &= \gamma_o^{ij}\, \gamma_{ij} \\
I_2 &= \gamma_o^{ri}\, \gamma_{ij}\, \gamma_o^{js}\, \gamma_{rs} \qquad\qquad \text{(IV-10)} \\
I_3 &= \mathrm{Det}(\gamma_o^{-1}\, \gamma) \ .
\end{aligned}
$$

Hence, an equation similar to (IV-4) gives

$$\pi^{ji} = 2\frac{\partial F}{\partial I_1}\frac{\partial I_1}{\partial \gamma_{ij}} + 2\frac{\partial F}{\partial I_2}\frac{\partial I_2}{\partial \gamma_{ij}} + 2\frac{\partial F}{\partial I_3}\frac{\partial I_3}{\partial \gamma_{ij}} \ . \qquad \text{(IV-11)}$$

From (IV-10) one obtains

$$\frac{\partial I_1}{\partial \gamma_{ij}} = \gamma_o^{ij} \ , \qquad \text{(IV-12)}$$

$$\frac{\partial I_2}{\partial \gamma_{ij}} = 2\gamma_o^{js}\,\gamma_{sr}\,\gamma_o^{ri} \ , \qquad \text{(IV-13)}$$

$$\frac{\partial I_3}{\partial \gamma_{ij}} = \det \gamma_o^{-1} \frac{\partial}{\partial \gamma_{ij}}\, \gamma_{ij}\, \Gamma^{ji} = \Gamma^{ji} \qquad \text{(IV-14)}$$

where Γ^{ji} is the cofactor of the element γ_{ij} in the matrix γ. Since one also has the expression for the elements of the inverse matrix,

$$\gamma^{ij} = \frac{\Gamma^{ij}}{\det \gamma} \ , \qquad \text{(IV-15)}$$

equation (IV-14) reduces to

$$\frac{\partial I_3}{\partial \gamma_{ij}} = I_3 \gamma^{ji}. \tag{IV-16}$$

Substituting (IV-12), (IV-13) and (IV-16) into (IV-11) one obtains

$$\pi^{ji} = 2\frac{\partial F}{\partial I_1}\gamma_o^{ij} + 2I_3\frac{\partial F}{\partial I_3}\gamma^{ij} + 4\frac{\partial F}{\partial I_2}\gamma_o^{js}\gamma_{sr}\gamma_o^{ri} \tag{IV-17}$$

or in a matrix notation

$$\pi = 2\frac{\partial F}{\partial I_1}\gamma_o^{-1} + 2I_3\frac{\partial F}{\partial I_3}\gamma^{-1} + 4\frac{\partial F}{\partial I_2}\gamma_o^{-1}\gamma\gamma_o^{-1}. \tag{IV-18}$$

Contrary to Lodge[(2)], one does not have a density factor in front of the right hand side of equation (IV-18) since one is using the free enrgy per unit volume instead of the free energy per unit mass.

Equation (IV-18) is rather general, and as was remarked in the paragraph below equation (IV-9) it does not include higher order terms because of the Hamilton-Cayley Theorem.

Equation (IV-18) can be used to represent the constitutive equation for a rubber-like material. In such a material one can assume incompressibility; hence, from (II-9), (II-10) and the last equation of equations (IV-10) one gets $I_3 = 1$. For a rubber-like material one can assume that F is independent of I_2; therfore, the last term of the right hand side of equation (IV-18) is zero. Since $I_3 = 1$ the coefficient of γ^{-1} is undetermined, one can call it p. It will be determined by the condition of static equilibrium of the body for elastostatics or by the equations of motion (III -9) for a dynamical system. A further simplification is to assume that $2\frac{\partial F}{\partial I_1}$ is a constant μ_0. Hence, one obtains the constitutive equation for a rubber-like incompressible material

$$\pi = -p\,\gamma^{-1} + \mu_0\,\gamma_0^{-1}\,. \qquad \text{(IV-19)}$$

Since $I_3 = 1$

$$\text{Det}\,\gamma = \text{Det}\,\gamma_0. \qquad \text{(IV-19-1)}$$

Equation (IV-19-1) is to be used jointly with the constitutive equation (IV-19).

Consider another approach similar to the one used to derive equation (IV-9) but more closely resembling the theory of linear elasticity. It consists of expanding the free energy in a Taylor's series in the strain tensor. One can expand the free energy up to third order, but the method could be generalized to higher orders. It is believed that since the third order terms will give rise to second order terms in the stress versus strain relationship this is sufficient to describe the behavior of many elastic solids.

As mentioned before one needs only to consider the strain part of the free energy. Following Landau[7] one can write the free energy as

$$F = \frac{1}{2}\left(\mu\,\epsilon^i_j\epsilon^j_i + \frac{\lambda}{2}(\epsilon^i_i)^2 + \frac{A}{3}\epsilon^i_k\,\epsilon^k_l\,\epsilon^l_i + B\epsilon^i_k\,\epsilon^k_i\,\epsilon^l_l + \frac{C}{3}(\epsilon^i_i)^3\right). \qquad \text{(IV-20)}$$

Expression (IV-20) is a generalization to mixed tensors of a Taylor's series. No linear terms are included since one assumes that the strain free state is also a stress free state. The factors μ, λ, A, B, C are assumed constants μ and λ are the Lamé parameters of classical elasticity. A, B, C are three constants that can be called anharmonic constants since the last three terms of the right hand side of equation (IV-20) resemble the anharmonic terms in the potential in terms of displacements for mechanics problems. These constants are also called the third order elastic constants or the third order elastic moduli. One can follow the same proccedure as the one used to derive equation (IV-9). Using equation (III-27), (IV-5) and (IV-6) after some algebra one obtains the equation

$$
\begin{aligned}
\pi = {} & \mu\, \gamma_0^{-1}(\gamma-\gamma_0)\gamma_0^{-1} \\
& + \frac{\lambda}{2}\mathrm{Trace}(\gamma_0^{-1}(\gamma-\gamma_0))\gamma_0^{-1} \\
& + A\gamma_0^{-1}(\gamma-\gamma_0)\gamma_0^{-1}(\gamma-\gamma_0)\gamma_0^{-1} \\
& + B\mathrm{Trace}(\gamma_0^{-1}(\gamma-\gamma_0)\gamma_0^{-1}(\gamma-\gamma_0))\gamma_0^{-1} \\
& + 2B\mathrm{Trace}(\gamma_0^{-1}(\gamma-\gamma_0))\gamma_0^{-1}(\gamma-\gamma_0)\gamma_0^{-1} \\
& + C(\mathrm{Trace}(\gamma_0^{-1}(\gamma-\gamma_0)))^2\gamma_0^{-1} \ .
\end{aligned}
\tag{IV-21}
$$

One should notice several things about this constitutive equation. From its symmetric form and from the symmetry of γ_0, $\gamma_0-\gamma$ one obtains the symmetry of the stress tensor π . This must be a necessary condition for any constitutive equation for finite displacements.

The first two terms of the right hand side of the equation resemble the terms in the classical theory of elasticity. The last four are the anharmonic contributions. The B term in the free energy give two terms of second order in γ_0 $(\gamma-\gamma_0)$ in the constitutive equation.

Equation (IV-21) will be our starting equation for problems of finite elasticity. Another way to derive equation (IV-21) is to start with an expression for the free energy in terms of the invariants J_1, J_2, J_3 of (IV-1) and use equation (IV-9). By starting with the free energy expression

$$F = \frac{1}{2}\left(\frac{\lambda}{4} J_1^2 + BJ_1J_2 + \frac{C}{3} J_1^3 + \frac{\mu}{2} J_2 + \frac{A}{3} J_3\right) \qquad \text{(IV-22)}$$

one obtains the same equation (IV-21). This approach, although giving rise to much less algebra hides, somehow the physical meanning of the different terms in the free energy.

One can examine equation (IV-21) in terms of a space coordinate system using an isomorphism at time t between the body and space coordinate system. Futhermore, to simplify things one can use an orthonormal coordinate system. Hence, one needs not differentiate between covariant and contravariant tensor components in this system since

$$g_{ij} = g^{ij} = \delta_i^j \qquad \text{(IV-23)}$$

or

$$g = I \qquad \text{(IV-23')}$$

using a matrix notation, where I is the unit matrix.

For the isomorphism at time t between the body and the space coordinate system π reduces to p and γ_0^{-1} to $\bar{B}$ which is related to the Finger strain tensor (Lodge[2] and Freed [5]) γ_0 reduces to $\bar{B}^{-1}$ since γ_0 is the inverse of γ_0^{-1} .Equation (IV-21) takes the form

$$p = \mu\bar{B}(I-\bar{B}^{-1})\bar{B} + \frac{\lambda}{2}\mathrm{Trace}(\bar{B}(I-\bar{B}^{-1}))\bar{B} + A\bar{B}(I-\bar{B}^{-1})\bar{B}(I-\bar{B}^{-1})\bar{B}$$
$$+ B\mathrm{Trace}(\bar{B}(I-\bar{B}^{-1})\bar{B}(I-\bar{B}^{-1}))\bar{B} + 2B\mathrm{Trace}(\bar{B}(I-\bar{B}^{-1}))\bar{B}(I-\bar{B}^{-1})\bar{B}$$
$$+ C(\mathrm{Trace}(\bar{B}(I-\bar{B}^{-1})))^2\bar{B} .$$

(IV-24)

The above equation, valid in an orthonomal space coordinate system, can easily be generalized to any space coordinate system by simply replacing I by the matrix g this matrix being made up of the covariant elements (g_{ij}). Keep in mind that p is made up of contravariant elements p^{ij}.

Going back to an orthonormal space coordinate system one can evaluate $\bar{B}^{-1}$. From equation (II-25) and $S^i = x^i$ one obtains

$$\bar{B}^{-1}_{ij} = \gamma_{ij}(\hat{x^i}, t_0) = g_{mn} \frac{\partial r^m}{\partial x^i} \frac{\partial r^n}{\partial x^j} \quad . \qquad \text{(IV-25)}$$

Remembering that r^i are the coordinates of a particle A at time t_0 and x^i its coordiantes at time t one obtains

$$x^i = r^i + u^i. \qquad \text{(IV-26)}$$

u^i are the components of the displacement; hence, from (IV-26)

$$\frac{\partial r^m}{\partial x^i} = \delta_m^i - \frac{\partial u^m}{\partial x^i} \quad . \tag{IV-27}$$

Substituting (IV-27) into (IV-25) and making use of (IV-23) one obtains

$$\bar{B}_{ij}^{-1} = \delta_i^j - \frac{\partial u^i}{\partial x^j} - \frac{\partial u^j}{\partial x^i} + \frac{\partial u^m}{\partial x^i}\frac{\partial u^m}{\partial x^j} \quad . \tag{IV-28}$$

Note, our $\bar{B}^{-1}$ is $C_t(x,t_0)C$ of Freed [5] with y^i replaced by x^i. Following a similar procedure one can obtain $\bar{B}$ or $\underline{B}_t(x,t_0)C$ of Freed. It is given by

$$\bar{B}_{ij} = \delta_i^j + \frac{\partial u^i}{\partial r^j} + \frac{\partial u^j}{\partial r^i} + \frac{\partial u^i}{\partial r^k}\frac{\partial u^j}{\partial r^k} \quad . \tag{IV-29}$$

If one substitutes (IV-28) and (IV-29) into (IV-24), one would obtain the stress tensor in the cartesian reference frame. Owing to the complexity of (IV-28) and (IV-29) the resulting equations would be rather intractable; hence, the benefit of using a body formalism for finite displacements.

In the limit of small displacements one can neglect second order terms in (IV-28) and (IV-29) and also in (IV-29) and take derivatives with respect to x^i instead of r^i. Hence, one can approximate $\bar{B}_{ij}$ by

$$\bar{B}_{ij} = \delta_i^j + \frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i} \quad . \tag{IV-30}$$

If one uses the infinitesimal strain tensor of classical elasticity

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i}\right) , \tag{IV-31}$$

one obtains by substituting (IV-30) into (IV-24) and looking only at the first two terms of (IV-24) for small displacements

$$p = \mu\ 2e(2e + I) + \frac{\lambda}{2}\,\mathrm{Trace}(2e)(2e + I). \tag{IV-32}$$

To first order it gives the well known stress strain relation of classical elasticity

$$p = 2\mu e + \lambda \mathrm{Trace}(e)I. \tag{IV-33}$$

In component form it becomes

$$p_{ij} = 2\mu\epsilon_{ij} + \lambda\ \epsilon_{ii}\delta_{ij} . \tag{IV-34}$$

Since one obtained (IV-34) from the body coordinate system equation (IV-21), this equation gives the correct result for infinitesimal displacements.

CHAPTER V

TORSION OF A RIGHT CIRCULAR CYLINDER

A. GENERAL CONSIDERATIONS

One can apply the body coordinate formalism to the torsion of a right circular cylinder the torsion being such that the deformations are finite. Let R be the radius of the original cylinder and ϕ_2 the torsion angle. Consider an $x\ y\ z$ coordinate system with the cylinder of length L fixed at the end z=0, the z axis coinciding with the axis of the cylinder (Fig (6)).

One can consider a simple system where the z=L end of the cylinder is twisted through an angle $\phi_2 L$ by an applied torque M. At the same time a force F is applied to the same end in such a way that the z=L surface parallel to the xoy plane stays the same. Hence, all the z=constant planes will stay the same although they will be rotated by angles $\phi_2 z$.

The body formalism, as one can see in Chapter II, besides a choice of a body coordinate system requires the use of the constitutive equation (IV-21) and the solution of the equilibrium equation (III-12). One has basically two choices for the body coordinate system. One which is isomorphic to a space coordinate system at time t_0 and the other one isomorphic to the space coordinate system at time t. Let us first look at the former. One can interchangibly use either the parameter ϕ_2 or the parameter t to represent the state of the system. t_0 is the state with $\phi_2=0$ which one will assume unstressed, and t the state corresponding to ϕ_2 and hence z=constant planes twisted by angles $\phi_2 z$.

If one uses an isomorphism of the body coordinate system at time t_0, then the coordinates S^1, S^2, S^3 will be equal to the cylindrical coordinate system ϕ_0 , z_0, r_0 of the xyz axis of fig (6). Hence, $\gamma_0 = g\ (\phi,\ r,\ z)$ has a simple form. This is helpful to use in the constitutive equation (Equation (IV-21)), and therefore, the calculation of π is reasonably simple. The calculation of γ from (II-27) is more complex. The condition of equilibrium π^{ij} =0 then involves the calculations of the Christoffel symbols of the second kind . See equation (III-10). From equations (II-28) and (II-29) one can calculate these elements. Since many of the γ_{ij} are not zero, many of the will not be zero, and the calculation of the divergence of π from (III-10) will be difficult. Since this will lead to a complex non-linear second order degree differential equation, it is better to take an isomorphism at time t. The calculation of the stress tensor π will be somehow more difficult than the calculation for the isomorphism at time t_0, but the calculation of π^{ij} much more tractable. Another advantage of the isomorphism at time t is that the direction of the normals to the z=L plane will have a simpler expression in terms of the basis vectors, and hence from (II-31) the tractions at the z=L end will be easier to calculate.

For an isomorphism at time t the body coordinate system S^1, S^2, S^3 will be equal to the cylindrical coordinate system ϕ,z,r. Let ϕ_0, z_0, r_0 be the space coordinates of a point A at time

t_o, Lagrangian coordinates, and the space coordinate system at time t are ϕ, z, r.

From the assumptions of the previous page one obtains the following relationships

$$\phi = \phi_o + \phi_2 z \ , \tag{V-1}$$

$$r = r(r_o), \text{ and} \tag{V-2}$$

$$z = \text{constant} = z_o. \tag{V-3}$$

In equation (V-2) one has made the assumption that the changes in radii are independent of z.

The inverse of the above equations are

$$\phi_o = \phi - \phi_2 z, \tag{V-4}$$

$$r_o = r_o\ (r), \text{ and} \tag{V-5}$$

$$z_o = z. \tag{V-6}$$

Hence, from the isomorphism of (ϕ, z, r) and S^1, S^2, S^3 one obtains

$$\phi_o = S^1 - \phi_2 S^2, \tag{V-7}$$

$$r_o = r_o\ (S^3), \text{ and} \tag{V-8}$$

$$z_o = S^2. \tag{V-9}$$

Equations (V-7), (V-8), (V-9) are the equivalent of equation (II-24), the parameter t being replaced by the parameter ϕ_2 which is the angle of torsion. Equations (V-7),(V-8),(V-9) are, therefore, the basic equations which describe the twisting of the right circular cylinder. One should keep in mind that the body coordinates S^1, S^2, S^3 of a point A are fixed quantities. Also in (V-8) $r_0 = r_0\ (S^3)$ is a short notation for r_0 is a function of the variable S^3.

It is interesting to look at the physical meanning of the transformation equations. The best way is to look at equations (V-1), (V-2), and (V-3). Let us consider a small element in a cylindrical coordinate system (fig (7)). The undeformed volume element being ABCDEFGH. After deformation it becomes A'B'C'D'E'F'G'H'. Several things happen. First, because of equation (V-2) the planes ABCD and EFGH have moved along the r direction. The motion is not a simple dilatation or contraction of the form r = (constant) r_0 but a general single valued transformation with well defined inverse, equation (V-5). Equation (V-3) means that the planes AEHD and CBFG, although rotated, keep a constant separation h (fig (7)) and equation (V-1) means that the surfaces AEHD and CBFG have slipped with respect to each other through an angle $d\phi = \phi_2 h$ (fig (7)). Hence, the transformations (V-1), (V-2), (V-3) resemble a shear; however, because of (V-2) it is not a constant volume process, and there is a change of volume between the elements ABCDEFGH and A'B'C'D'E'F'G'H'.

The space metric at time t in the cylindrical coordinate system is

$$g = \begin{pmatrix} r^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{(V-10)}$$

with the inverse

$$g^{-1} = \begin{pmatrix} \frac{1}{r^2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \qquad \text{(V-11)}$$

Because of the isomorphism at time t between the body and the space coordinate system one obtains from equation (II-21)

$$\gamma = \begin{pmatrix} (S^3)^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \gamma^{-1} = \begin{pmatrix} \frac{1}{(S^3)^2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{(V-12)}$$

with

$$\mathrm{Det}\ \gamma = (S^3)^2 . \qquad \text{(V-12')}$$

The body metric tensor at time t_0 is given by equation (II-25). In our notation $\bar{r}^1 = \phi_0$, $\bar{r}^2 = z_0$, $\bar{r}^3 = r_0$. In these equations $\bar{r}^2$ is the notion for the coordinate $\bar{r}^2$ and not $\bar{r}$ squared, $\bar{r}^3$ is a notation for coordiante $\bar{r}^3$ and not $\bar{r}$ cubed.

Therfore,

$$\gamma_{11,0} = g_{11} \frac{\partial \bar{r}^1}{\partial S^1} \frac{\partial \bar{r}^1}{\partial S^1} + g_{22} \frac{\partial \bar{r}^2}{\partial S^1} \frac{\partial \bar{r}^2}{\partial S^1} + g_{33} \frac{\partial \bar{r}^3}{\partial S^1} \frac{\partial \bar{r}^3}{\partial S^1} . \qquad \text{(V-13)}$$

From equations (V-7), (V-8), (V-9) one obtains

$$\gamma_{11,0} = r_0^2 \qquad \text{(V-14)}$$

where in the above equation it is understood that r_o is a function of S^3 and ϕ_2.

One follows the same procedure to calculate all the γ_{ij} and the body metric tensor γ written in matrix form becomes

$$\gamma_o = \begin{pmatrix} r_o^2 & -r_o^2\phi_2 & 0 \\ -r_o^2\phi_2 & 1 + r_o^2\phi_2^2 & 0 \\ 0 & 0 & \left(\frac{\partial r_o}{\partial S^3}\right)^2 \end{pmatrix} . \tag{V-15}$$

From the above expression one can calculate the determinant of γ_o.

$$\det \gamma_o = \left(\frac{\partial r_o}{\partial S^3}\right)^2 r_o^2 , \tag{V-16}$$

and with the help of (V-16) one gets

$$\gamma_o^{-1} = \begin{pmatrix} \frac{1}{r_o^2} + \phi_2^2 & \phi_2 & 0 \\ \phi_2 & 1 & 0 \\ 0 & 0 & \frac{1}{\left(\frac{\partial r_o}{\partial S^3}\right)^2} \end{pmatrix} . \tag{V-17}$$

It is interesting to interpret the equations (V-12) and (V-17) in a geometrical manner. One starts with a twisted cubic element similar to the one at time t of figure (7) but at time t_o. One ends up with a simple cubic element at time t.

Since equations (II-6) and (II-7) are valid, with the help of fig (8) it is obvious that in the initial state at time t_o $\vec{e}_{1,o}.\vec{e}_{3,o}=0$, $\vec{e}_{2,o}.\vec{e}_{3,o}=0$, and only the non-diagonal elements $\vec{e}_{1,o}.\vec{e}_{2,o}$ are non-zero. In the state t since the basis vectors are orthogonal all the non-diagonal elements of γ are zero ($\vec{e}_i.\vec{e}_j=0$ if $i \neq j$).

Given the body metric tensor γ and a certain constitutive equation one can calculate the stress tensor. The condition of equilibrium being equation (III-12) one has to calculate the divergence of this stress tensor in the body coordinate system at time t, and hence because of equation (III-10), one has to calculate the Christoffel symbols of the second kind in this coordinate system.

Because of (V-12) the only non zero partial derivative of the body tensor metric component is $\frac{\partial \gamma_{11}}{\partial S^3} = 2S^3$;hence, from (II-28) the only non-zero Christoffel symbols of the first kind are

$$\Gamma_{113} = S^3 \qquad \Gamma_{131} = S^3 \qquad \Gamma_{311} = -S^3 . \tag{V-18}$$

From equation (II-29) the only non-zero Christoffel symbols of the second kind are

$$\Gamma^3_{11} = -S^3 \qquad \Gamma^1_{13} = \Gamma^1_{31} = \frac{1}{S^3} \quad . \tag{V-19}$$

Hence, from (III-10) one obtains the following equations;

$$\frac{\partial \Pi^{11}}{\partial S^1} + \frac{\partial \Pi^{12}}{\partial S^2} + \frac{\partial \Pi^{13}}{\partial S^3} = 0 \quad , \tag{V-20}$$

$$\frac{\partial \pi^{21}}{\partial s^1} + \frac{\partial \pi^{22}}{\partial s^2} + \frac{\partial \pi^{23}}{\partial s^3} = 0 \qquad \text{(V-21)}$$

and

$$\frac{\partial \pi^{31}}{\partial s^1} + \frac{\partial \pi^{32}}{\partial s^2} + \frac{\partial \pi^{33}}{\partial s^3} - \pi^{11} s^3 + \frac{\pi^{33}}{s^3} = 0 \quad . \qquad \text{(V-22)}$$

Although (V-20) and (V-21) resemble the divergence equations in an orthogonal coordinate system, equation (V-22) contains extra terms because of the use of a curvilinear coordinate system.

Besides conditions (V-20), (V-21), (V-22) which must be satisfied by the stress tensor, the traction on the lateral cylindrical surface must also vanish.

One starts with equation (II-31) with the normal vector being specified by equations (II-32) and (II-33).

The lateral cylindrical surface corresponds to the equation S^3=constant and from (II-32) has only a component in the $\hat{r}$ direction. Also with the help of equation (II-32) and (II-12) one finds that the coefficients of the $\hat{r}$ component are one. Therfore,

$$\nu_1 = 0 \qquad \nu_2 = 0 \qquad \nu_3 = 1 \ . \qquad \text{(V-23)}$$

Making use of (II-31) one gets

$$f^i = \pi^{i3} \ . \qquad \text{(V-24)}$$

In the calculations that are to follow one will find that $\pi^{13} = \pi^{23} = 0$ and equation (V-24) can be reduced to

$$f^3 = \pi^{33}\Big)_{S_0^3} = 0 \; . \qquad \text{(V-25)}$$

In equation (V-25) the stress component π^{33} is evaluated at $S^3 = S^3{}_o$. $S^3{}_o$ is the value of S^3 cooresponding to the value of $r_o = R$ where R is the radius of the cylinder in the undeformed state.

One is also interested in calculating the traction at the z=L end of the cylinder. In this case the equation of the surface corresponds to S^2=constant. Therfore, from equations (II-32) and (II-33) with the help of (V-12') one obtains

$$\nu_1 = 0 \qquad \nu_2 = 1 \qquad \nu_3 = 0 \; . \qquad \text{(V-26)}$$

Equation (V-26) is substituted into (II-31) which gives

$$f^i = \pi^{i2} \qquad \text{(V-27)}$$

As one will see later π^{32} is zero and hence from the above equation one obtains

$$\vec{f} = \pi^{12}\,\vec{e}_1 + \pi^{22}\,\vec{e}_2 . \qquad \text{(V-28)}$$

Because of equation (II-7) and equation (V-12)

$$\vec{e}_1 = \hat{\varphi}\ S^3 \qquad \vec{e}_2 = \hat{z}\ .$$

equation (V-28) reduces to

$$\vec{f} = \pi^{12}\ S^3 \hat{\varphi} \quad + \pi^{22}\ \hat{z} \tag{V-29}$$

where $\hat{\varphi}$, $\hat{z}$ are the unit vectors in the ϕ and z directions (fig 7).

From (V-29) one can calculate the applied torque at the end surface and the total force applied at the end surface

$$M = 2\pi \int_0^{S_0^3} \Pi^{12}\,(S^3)^3\, dS^3 \tag{V-30}$$

and

$$F = 2\pi \int_0^{S_0^3} \Pi^{22}\, S^3 dS^3 \tag{V-31}$$

where in the factor 2π , $\pi = 3.14$, is not to be confused with the stress tensor.

B. Rubber Like Solids

The constitutive equations are given by equations (IV-19) and (IV-19') or

$$\pi = -p\ \gamma^{-1} + \mu_0\ \gamma_0^{-1} \tag{V-32}$$

and

$$\text{Det}\ \gamma = \det\ \gamma_0 \tag{V-33}$$

where γ, γ_0, γ^{-1}, γ_0^{-1} are given by equations (IV-12), (IV-15) and (IV-17). From equations (V-12') and (V-16), and equation (V-33) one obtains

$$\left(\frac{\partial r_0}{\partial S^3}\right)^2 r_0^2 = (S^3)^2. \tag{V-34}$$

Upon integration of the above equation one obtains

$$S^3 = \sqrt{r_0^2 + C_1} \tag{V-35}$$

where C_1 is a constant. Since one is twisting a solid cylinder, the axis of this cylinder is undeformed in the process. Points with coordinates $r_0=0$ must have new coordinates $S^3=0$; hence, one must have $C_1=0$ in (V-35). Therefore,

$$S^3 = r_0. \tag{V-36}$$

The above result is to be expected because the volume of the cylinder is constant and the end is held at a constant distance so that the radius of the cylinder must be unchanged.

From (V-32) with the use of (V-12) and (V-17) one gets

$$\Pi = \begin{pmatrix} (\mu_0 - p)\frac{1}{(S^3)^2} + \mu_0 \phi_2^2 & \mu_0 \phi_2 & 0 \\ \mu_0 \phi_2 & \mu_0 - p & 0 \\ 0 & 0 & \mu_0 - p \end{pmatrix} . \quad \text{(V-37)}$$

Substituting this equation into (V-20), (V-21), (V-22) one obtains

$$\frac{\partial p}{\partial S^1} = 0 \qquad \frac{\partial p}{\partial S^2} = 0 \qquad \frac{\partial}{\partial S^3}(\mu_0 - p) - \mu_0 S^3 \phi_2^2 = 0 . \quad \text{(V-38)}$$

Integrating the above equations one obtains

$$p = -\mu_0 \frac{(S^3)^2}{2} \phi_2^2 + p_0 \quad \text{(V-39)}$$

where p_0 is a constant. To determine this constant one can use equation (V-25) and obtain

$$\mu_0 - p = \mu_0 \frac{\phi_2^2}{2}((S^3)^2 - R^2) . \quad \text{(V-40)}$$

As expected $\mu_0 - p$ is proportional to ϕ_2. When $\phi_2 = 0$ which is the unstressed state $\mu_0 - p = 0$, and hence from (V-37) $\pi = 0$.

From (V-30) and (V-37) one obtains

$$M = 2\pi \int_0^{S_0^3} \mu_0 \phi_2 (s^3)^3 ds^3 \qquad \text{(V-41)}$$

Since $r_0 = S^3$ and $S_0^3 = R$, it follows that

$$M = \pi \mu_0 \phi_2 \frac{R^4}{2} . \qquad \text{(V-42)}$$

From (V-31), (V-37), and (V-4) one obtains

$$F = 2\pi \int_0^{S_0^3} \mu_0 \frac{\phi_2^2}{2} \left((s^3)^2 - R^2\right) s^3 ds^3 \qquad \text{(V-43)}$$

or

$$F = - \pi \mu_0 \phi_2^2 \frac{R^4}{4} . \qquad \text{(V-44)}$$

Equation (V-42) gives the same result as the one from the classical theory of elasticity for small deformations, if one takes the constant μ_0 to be μ. As in the classical theory of elasticity the torque is proportional to the torsion angle ϕ_2.

The force F, if one assumes μ_0 to be μ is proportional to the square of the dimensionless parameter $\phi_2 R$. Hence, for small deformations it can be neglected. For finite deformations its expression (V-44) is rather simple. The minus sign implies that the end of the cylinder has to be compressed to keep the length of the cylinder constant.

C. Elastic Solids

The constitutive equation for an elastic solid with large deformations is equation (IV-21).

This constitutive equation is written in terms of the body metric tensors γ_0 and γ . The body metric tensors are evaluated at time t_0 when the torsion angle $\phi_2 = 0$ and at time t when the torsion angle is ϕ_2 . The parameters entering the constitutive equations are the Lame coefficients μ and λ and the third order elastic moduli A, B, and C. As mentioned previously the contitutive equation is valid for a compressible solid.

One starts with γ_0, γ_0^{-1}, γ, γ^{-1} given by equations (V-15), (V-17) and (V-12) and obtains

$$\gamma_0^{-1}(\gamma-\gamma_0) = \begin{pmatrix} \left(\frac{s^3}{r_0}\right)^2 - 1 + \phi_2^2 (s^3)^2 & \phi_2 & 0 \\ \phi_2 (s^3)^2 & 0 & 0 \\ 0 & 0 & \frac{1}{\left(\frac{\partial r_0}{\partial s^3}\right)^2} - 1 \end{pmatrix} . \quad \text{(V-45)}$$

Hence from the above

$$\alpha = \mathrm{Trace}\left(\gamma_0^{-1}(\gamma-\gamma_0)\right) = \left(\frac{s^3}{r_0}\right)^2 - 2 + \phi_2^2 (s^3)^2 + \frac{1}{\left(\frac{\partial r_0}{\partial s^3}\right)^2} \quad . \quad \text{(V-46)}$$

One also gets

$$\gamma_0^{-1}(\gamma-\gamma_0)\gamma_0^{-1} = \begin{pmatrix} \left(\frac{s^3}{r_0}\right)^2\frac{1}{r_0^2} - \frac{1}{r_0^2} + 2\phi_2^2\left(\frac{s^3}{r_0}\right)^2 + \phi_2^4 (s^3)^2 & \phi_2\left(\frac{s^3}{r_0}\right)^2 + \phi_2^3 (s^3)^2 & 0 \\ \phi_2\left(\frac{s^3}{r_0}\right)^2 + \phi_2^3 (s^3)^2 & \phi_2 (s^3)^2 & 0 \\ 0 & 0 & \frac{1}{\left(\frac{\partial r_0}{\partial s^3}\right)^2}\left(\frac{1}{\left(\frac{\partial r_0}{\partial s^3}\right)^2} - 1\right) \end{pmatrix} \qquad \text{(V-47)}$$

and

$$\gamma_0^{-1}(\gamma-\gamma_0)\gamma_0^{-1}(\gamma-\gamma_0) = \begin{pmatrix} \left[\left(\frac{s^3}{r_0}\right)^2 - 1 + \phi_2^2 (s^3)^2\right]^2 + \phi_2^2 (s^3)^2 & \left[\left(\frac{s^3}{r_0}\right)^2 - 1 + \phi_2^2 (s^3)^2\right]\phi_2 & 0 \\ \left[\left(\frac{s^3}{r_0}\right)^2 - 1 + \phi_2^2 (s^3)^2\right]\phi_2 (s^3)^2 & \phi_2^2 (s^3)^2 & 0 \\ 0 & 0 & \left(\frac{1}{\left(\frac{\partial r_0}{\partial s^3}\right)^2} - 1\right)^2 \end{pmatrix}. \qquad \text{(V-48)}$$

Hence from the above

$$d = \mathrm{Trace}\left(\gamma_0^{-1}(\gamma-\gamma_0)\gamma_0^{-1}(\gamma-\gamma_0)\right) = \left(\left(\frac{s^3}{r_0}\right)^2 - 1 + \phi_2^2 (s^3)^2\right)^2 + 2\phi_2^2 (s^3)^2 + \left(\frac{1}{\left(\frac{\partial r_0}{\partial s^3}\right)^2} - 1\right)^2. \qquad \text{(V-49)}$$

The last expression needed is

$$\gamma_0^{-1}(\gamma-\gamma_0)\gamma_0^{-1}(\gamma-\gamma_0)\gamma_0^{-1} = \begin{pmatrix} c_{11} & c_{12} & 0 \\ c_{21} & c_{22} & 0 \\ 0 & 0 & c_{33} \end{pmatrix} \qquad \text{(V-50)}$$

with

$$c_{12} = \left(\left(\left(\frac{s^3}{r_0}\right)^2 - 1 + \phi_2^2 (s^3)^2\right)^2 + \phi_2^2 (s^3)^2\right)\phi_2 + \left(\left(\frac{s^3}{r_0}\right)^2 - 1 + \phi_2^2 (s^3)^2\right)\phi_2 , \qquad \text{(V-51)}$$

$$c_{21} = \left(\left(\frac{s^3}{r_0}\right)^2 - 1 + \phi_2^2 (s^3)^2\right)\phi_2^2 (s^3)^2\left(\frac{1}{r_0^2} + \phi_2^2\right) + \phi_2^2 (s^3)^2 \phi_2 , \qquad \text{(V-52)}$$

$$c_{22} = \left(\left(\frac{s^3}{r_0}\right)^2 - 1 + \phi_2^2 (s^3)^2\right)\phi_2^2 (s^3)^2 + \phi_2^2 (s^3)^2, \qquad \text{(V-52')}$$

$$c_{11} = \left(\left(\frac{s^3}{r_0}\right)^2 - 1 + \phi_2^2 (s^3)^2\right)^2 \left(\frac{1}{r_0^2} + \phi_2^2\right) + \left(\left(\frac{s^3}{r_0}\right)^2 - 1 + \phi_2^2 (s^3)^2\right)\phi_2^2 \qquad \text{(V-53)}$$

and

$$c_{33} = \frac{1}{\left(\frac{\partial r_0}{\partial s^3}\right)^2}\left(\frac{1}{\left(\frac{\partial r_0}{\partial s^3}\right)^2} - 1\right)^2. \qquad \text{(V-54)}$$

After some algebraic manipulations one can show that $c_{12} = c_{21}$ as expected. Substituting equations (V-45) through equations (V-54) into the constitutive equation (IV-21) one obtains

$$\Pi = \mu \begin{pmatrix} \left(\frac{s^3}{r_0}\right)^2 \frac{1}{r_0^2} - \frac{1}{r_0^2} + 2\phi_2^2 \left(\frac{s^3}{r_0}\right)^2 + \phi_2^4 (s^3)^2 & \phi_2 \left(\frac{s^3}{r_0}\right)^2 + \phi_2^3 (s^3)^2 & 0 \\ \phi_2 \left(\frac{s^3}{r_0}\right)^2 + \phi_2^3 (s^3)^2 & \phi_2^2 (s^3)^2 & 0 \\ 0 & 0 & \frac{1}{\left(\frac{\partial r_0}{\partial s^3}\right)^2}\left(\frac{1}{\left(\frac{\partial r_0}{\partial s^3}\right)^2} - 1\right) \end{pmatrix}$$

$$+ \frac{\lambda}{2}\left(\left(\frac{S^3}{r_0}\right)^2 - 2 + \phi_2^2 (S^3)^2 + \frac{1}{\left(\frac{\partial r_0}{\partial S^3}\right)^2}\right)\begin{pmatrix} \frac{1}{r_0^2}+\phi_2^2 & \phi_2 & 0 \\ \phi_2 & 1 & 0 \\ 0 & 0 & \frac{1}{\left(\frac{\partial r_0}{\partial S^3}\right)^2} \end{pmatrix}$$

$$+ A \begin{pmatrix} c_{11} & c_{12} & 0 \\ c_{21} & c_{22} & 0 \\ 0 & 0 & c_{33} \end{pmatrix} + 2Bd \begin{pmatrix} \frac{1}{r_0^2}+\phi_2^2 & \phi_2 & 0 \\ \phi_2 & 1 & 0 \\ 0 & 0 & \frac{1}{\left(\frac{\partial r_0}{\partial S^3}\right)^2} \end{pmatrix}$$

$$+ B\left(\left(\frac{S^3}{r_0}\right)^2 - 2 + \phi_2^2 (S^3)^2 + \frac{1}{\left(\frac{\partial r_0}{\partial S^3}\right)^2}\right)\begin{pmatrix} \left(\frac{S^3}{r_0}\right)^2\frac{1}{r_0^2} - \frac{1}{r_0^2} + 2\phi_2^2\left(\frac{S^3}{r_0}\right)^2 + \phi_2^4 (S^3)^2 & \phi_2\left(\frac{S^3}{r_0}\right)^2 + \phi_2^3 (S^3)^2 & 0 \\ \phi_2\left(\frac{S^3}{r_0}\right)^2 + \phi_2^3 (S^3)^2 & \phi_2^2 (S^3)^2 & 0 \\ 0 & 0 & \frac{1}{\left(\frac{\partial r_0}{\partial S^3}\right)^2}\left(\frac{1}{\left(\frac{\partial r_0}{\partial S^3}\right)^2} - 1\right) \end{pmatrix}$$

$$+ C\left(\left(\frac{S^3}{r_0}\right)^2 - 2 + \phi_2^2 (S^3)^2 + \frac{1}{\left(\frac{\partial r_0}{\partial S^3}\right)^2}\right)^2 \begin{pmatrix} \frac{1}{r_0^2}+\phi_2^2 & \phi_2 & 0 \\ \phi_2 & 1 & 0 \\ 0 & 0 & \frac{1}{\left(\frac{\partial r_0}{\partial S^3}\right)^2} \end{pmatrix} . \qquad \text{(V-55)}$$

The equations of equilibrium (V-20), (V-21) involve the derivatives of π^{11}, π^{12}, π^{21}, π^{22}, with respect to S^1 and S^2. Since these components are only functions of S^3, the partial derivatives are all zero. Also from (V-55) one notices that π^{13} and π^{23} are zero; hence, equations (V-20) and (V-21) are automatically satisfied by the stres tensor π given by equation (V-55). Equation (V-22) reduces to

$$\frac{\partial \pi^{33}}{\partial S^3} - \pi^{11} S^3 + \frac{\pi^{33}}{S^3} = 0 \qquad \text{(V-56)}$$

or

$$\frac{\partial}{\partial S^3}\left(S^3 \pi^{33}\right) - \pi^{11}(S^3)^2 = 0 \quad , \qquad \text{(V-57)}$$

making use of equation (V-55) and (V-57) one obtains;

$$\frac{\partial}{\partial S^3} S^3 \Bigg\{ \mu \frac{1}{\left(\frac{\partial r_o}{\partial S^3}\right)^2}\left(\frac{1}{\left(\frac{\partial r_o}{\partial S^3}\right)^2} - 1\right) + \frac{\lambda}{2}\left(\left(\frac{S^3}{r_o}\right)^2 - 2 + \phi_2^2 (S^3)^2 + \frac{1}{\left(\frac{\partial r_o}{\partial S^3}\right)^2}\right)\frac{1}{\left(\frac{\partial r_o}{\partial S^3}\right)^2}$$

$$+ A\left(\frac{1}{\left(\frac{\partial r_o}{\partial S^3}\right)^2} - 1\right)^2 \frac{1}{\left(\frac{\partial r_o}{\partial S^3}\right)^2} + 2B\Bigg(\left(\left(\frac{S^3}{r_o}\right)^2 - 1 + \phi_2^2 (S^3)^2\right)^2 + 2\phi_2^2 (S^3)^2$$

$$+ \left(\frac{1}{\left(\frac{\partial r_o}{\partial S^3}\right)^2} - 1\right)^2\Bigg) \frac{1}{\left(\frac{\partial r_o}{\partial S^3}\right)^2}$$

$$+ B\left(\left(\frac{S^3}{r_o}\right)^2 - 2 + \phi_2^2 (S^3)^2 + \frac{1}{\left(\frac{\partial r_o}{\partial S^3}\right)^2}\right)\frac{1}{\left(\frac{\partial r_o}{\partial S^3}\right)^2}\left(\frac{1}{\left(\frac{\partial r_o}{\partial S^3}\right)^2} - 1\right)$$

(V-58)

$$+ C\left(\left(\frac{s^3}{r_0}\right)^2 - 2 + \phi_2^2 (s^3)^2 + \frac{1}{\left(\frac{\partial r_0}{\partial s^3}\right)^2}\right)^2 \frac{1}{\left(\frac{\partial r_0}{\partial s^3}\right)^2} \Bigg\}$$

$$- \Bigg\{ \mu\left(\left(\frac{s^3}{r_0}\right)^2\left(\frac{s^3}{r_0}\right)^2 - \left(\frac{s^3}{r_0}\right)^2 + 2\phi_2^2 (s^3)^2 \left(\frac{s^3}{r_0}\right)^2 + \phi_2^4 (s^3)^4\right)$$

$$+ \lambda/2 \left(\left(\frac{s^3}{r_0}\right)^2 - 2 + \phi_2^2 (s^3)^2 + \frac{1}{\left(\frac{\partial r_0}{\partial s^3}\right)^2}\right)\left(\left(\frac{s^3}{r_0}\right)^2 + \phi_2^2 (s^3)^2\right)$$

$$+ A\Bigg[\left(\left(\left(\frac{s^3}{r_0}\right)^2 - 1 + \phi_2^2 (s^3)^2\right)^2 + \phi_2^2 (s^3)^2\right)\left(\left(\frac{s^3}{r_0}\right)^2 + \phi_2^2 (s^3)^2\right)$$

$$+ \left(\left(\frac{s^3}{r_0}\right)^2 - 1 + \phi_2^2 (s^3)^2\right)\phi_2^2 (s^3)^2 \Bigg] +$$

$$2B\left(\left(\left(\frac{s^3}{r_0}\right)^2 - 1 + \phi_2^2 (s^3)^2\right)^2 + 2\phi_2^2 (s^3)^2 + \left(\frac{1}{\left(\frac{\partial r_0}{\partial s^3}\right)^2} - 1\right)^2\right)\left(\left(\frac{s^3}{r_0}\right)^2 + \phi_2^2 (s^3)^2\right)$$

$$+ B\left(\left(\frac{s^3}{r_0}\right)^2 - 2 + \phi_2^2 (s^3)^2 + \frac{1}{\left(\frac{\partial r_0}{\partial s^3}\right)^2}\right)\left(\left(\frac{s^3}{r_0}\right)^2\left(\frac{s^3}{r_0}\right)^2 - \left(\frac{s^3}{r_0}\right)^2 + 2\phi_2^2 (s^3)^2 \left(\frac{s^3}{r_0}\right)^2\right.$$

$$\left. + \phi_2^4 (s^3)^4\right)$$

$$+ C\left(\left(\frac{s^3}{r_0}\right)^2 - 2 + \phi_2^2 (s^3)^2 + \frac{1}{\left(\frac{\partial r_0}{\partial s^3}\right)^2}\right)^2\left(\left(\frac{s^3}{r_0}\right)^2 + \phi_2^2 (s^3)^2\right) \Bigg\} = 0 \; .$$

The above equation is a non-linear second order differential equation. The parameters in this equation besides μ, λ, A, B, C are the torsion angle squared and to the fourth power. Hence, in principle one can solve this equation and obtain $r_0 = r_0(S^3, \phi_2^2)$. In the above equation the partial derivative simply implies the derivative with respect to S^3 at constant ϕ_2 and could be written as a total derivative.

Although one will use equation (V-58) as such one can also rewrite equation (V-58) by considering S^3 to be a function of r_0, $S^3 = S^3(r_0, \phi_2^2)$. From (V-56), after a change of variable, and since π^{33} can be written as

$$\pi^{33} = X\left(\frac{\partial S^3}{\partial r_0}\right)^2 \tag{V-59}$$

one obtains

$$S^3\left(\frac{\partial X}{\partial r_0}\left(\frac{\partial S^3}{\partial r_0}\right) + 2X\frac{\partial^2 S^3}{\partial r_0^2}\right) - \pi^{11}(S^3)^2 + X\left(\frac{\partial S^3}{\partial r_0}\right)^2 = 0 \tag{V-60}$$

where

$$\begin{aligned} X = {} & \mu\left(\left(\frac{\partial S^3}{\partial r_0}\right)^2 - 1\right) + \frac{\lambda}{2}\left(\left(\frac{S^3}{r_0}\right)^2 - 2 + \phi_2^2(S^3)^2 + \left(\frac{\partial S^3}{\partial r_0}\right)^2\right) \\ & + A\left(\left(\frac{\partial S^3}{r_0}\right)^2 - 1\right) + 2B\left(\left[\left(\frac{S^3}{r_0}\right)^2 - 1 + \phi_2^2(S^3)^2\right]^2 + 2\phi_2^2(S^3)^2 + \left(\left(\frac{\partial S^3}{\partial r_0}\right)^2 - 1\right)^2\right) \\ & + B\left(\left(\frac{S^3}{r_0}\right)^2 - 2 + \phi_2^2(S^3)^2 + \left(\frac{\partial S^3}{\partial r_0}\right)^2\right)\left(\left(\frac{\partial S^3}{\partial r_0}\right)^2 - 1\right) \\ & + C\left(\left(\frac{S^3}{r_0}\right)^2 - 2 + \phi_2^2(S^3)^2 + \left(\frac{\partial S^3}{\partial r_0}\right)^2\right)^2 \end{aligned} \tag{V-61}$$

In equation (V-60) $-\pi^{11}\ (S^3)^2$ is given by the part of equation (V-58) on page 56 starting with μ . Here equation (V-60) is also a second order non-linear differential equation, but now S^3 is considered to be a function of the independent variable r_0.

One will start with equation (V-58). The first two terms of the solution of this equation in an expansion in powers of ϕ_2^2 are

$$r_0 = S^3 - \phi_2^2\, f - \phi_2^4\, g \qquad \text{(V-62)}$$

where f and g are two unknown functions of S^3. It is obvious from equation (V-58) that in the limit $\phi_2 \to 0$ the solution is $r_0 = S^3$ corresponding to the cylinder staying in its unstressed state.

Making use of equation (V-62) and keeping only terms up to order ϕ_2^4 one obtains

$$\left(\frac{S^3}{r_0}\right)^2 = 1 + 2\phi_2^2 \frac{f}{S^3} + \phi_2^4\left(\frac{2g}{S^3} + 3\left(\frac{f}{S^3}\right)^2\right) \qquad \text{(V-63)}$$

and

$$\frac{1}{\left(\frac{\partial r_0}{\partial S^3}\right)^2} = 1 + 2\phi_2^2 f' + \phi_2^4\left(2g' + 3f'^2\right) \qquad \text{(V-64)}$$

where for simplicity of notation

$$f' = \frac{\partial f}{\partial S^3} \qquad\qquad g' = \frac{\partial g}{\partial S^3} \qquad \text{(V-65)}$$

Substituting (V-63), (V-64) and (V-62) into (V-58) one obtains

$$
\begin{aligned}
&\frac{\partial}{\partial s^3} s^3 \{ \phi_2^2 [\mu f' + \lambda/2 (\frac{2f}{s^3} + 2f' + (s^3)^2)] + \phi_2^4 [\mu (2f')^2 + \mu (2g' + 3f'^2) \\
&+ \frac{\lambda}{2} (\frac{2g}{s^3} + 3 (\frac{f}{s^3})^2 + 2g' + 3f'^2) + \frac{\lambda}{2} 2f' (\frac{2f}{s^3} + 2f' + (s^3)^2) \\
&+ 4B \phi_2^2 (s^3)^2] + \phi_2^4 [4Af'^2 + 2B ((\frac{2f}{s^3} + (s^3))^2 + 4f'^2) + B (2f' (\frac{2f}{s^3} + 2f' + (s^3)^2) \\
&+ C (\frac{2f}{s^3} + 2f' + (s^3)^2) + 8Bf'(s^3)^2] \} - \{ \phi_2^2 [\mu (\frac{2f}{s^3} + 2(s^3)^2) \\
&+ \frac{\lambda}{2} ((\frac{2f}{s^3}) + 2f' + (s^3)^2)] + \phi_2^4 [\mu (7 (\frac{f}{s^3})^2 + \frac{2g}{s^3} + 4 (\frac{f}{s^3}) (s^3)^2 + (s^3)^4) \\
&+ \lambda/2 (((\frac{2f}{s^3}) + 2f' + (s^3)^2) (\frac{2f}{s^3} + (s^3)^2) + (\frac{2g}{s^3} + 3 (\frac{f}{s^3})^2 + 2g' + 3f'^2))] \\
&+ \phi_2^2 [4B(s^3)^2 + A(s^3)^2] + \phi_2^4 [A ((\frac{2g}{s^3} + 3 (\frac{f}{s^3})^2 + (\frac{2f}{s^3} + (s^3)^2)^2 \\
&+ (\frac{2f}{s^3} + (s^3)^2) (s^3)^2) + 2B ((\frac{2f}{s^3} + (s^3)^2)^2 + 4f'^2 + 2(s^3)^2 (\frac{2f}{s^3} + (s^3)^2)) \\
&+ B (\frac{2f}{s^3} + 2f' + (s^3)^2) (\frac{2f}{s^3} + 2(s^3)^2) + C (\frac{2f}{s^3} + 2f' + (s^3)^2)^2 \} .
\end{aligned}
$$

(V-66)

After expanding the derivatives in the above equation setting the coefficients of ϕ_2^2 and ϕ_2^4 equal to zero and considering first the coefficient of ϕ_2^2 one obtains

$$(s^3)^2 f'' + s^3 f' - f = -(s^3)^2 \left(\frac{\lambda - 2\mu - A + 4B}{\lambda + 2\mu} \right) . \tag{V-67}$$

This a linear second order differential equation. This is the same differential equations as Murnaghan's (9) equation.

The solution of the differential equation is

$$f = \frac{\tilde{A}}{s^3} + \tilde{B} s^3 + \tilde{C} (s^3)^3 \tag{V-68}$$

where $\tilde{A}$ and $\tilde{B}$ are constants to be determined from the boundry conditions and the constant $\tilde{C}$ is given by

$$\tilde{C} = -\frac{1}{8} \left(\frac{\lambda - 2\mu + 4B - A}{\lambda + 2\mu} \right) . \tag{V-69}$$

Since one is considering the deformation of a solid cylinder, one must set the coefficient of $\frac{1}{S^3}$ equal to zero since $r_0=0$ corresponds to $S^3=0$ so that

$$f = \tilde{B} s^3 + \tilde{C} (s^3)^3 . \tag{V-70}$$

The coefficient of ϕ_2^4 gives

$$(s^3)^2 g'' + s^3 g' - g = \frac{-L s^3}{\lambda + 2\mu} . \tag{V-71}$$

The differential equation for the second order correction, equation (V-71), has a left-hand side term similar to the left hand side terms of the differential equation for the first order correction, equation (V-67).

The right hand side of equation (V-71) can be rewritten as

$$L = -M - N(S^3)^2 - P(S^3)^4 \tag{V-72}$$

where M, N, P are constants. After some algebra one obtains from (V-66) making use of the first order term of equation (V-70);

$$N = \tilde{B}(\mu(112\tilde{C}-4) + \frac{\lambda}{2}98\tilde{C} + A(16\tilde{C}-8) + B(196\tilde{C}+28) + C(64\tilde{C}+8)) \tag{V-73}$$

and

$$P = \mu(304\tilde{C}^2-1) + \frac{\lambda}{2}(224\tilde{C}^2+5\tilde{C}-1) + B(118\tilde{C}^2+145\tilde{C}+2) - 4C(8\tilde{C}+1)^2 + A(176\tilde{C}^2-8\tilde{C}-3) \tag{V-74}$$

and

$$M = 0 \tag{V-75}$$

Since M = 0, the corresponding solution in equation (V-71) which would have been $S^3 \log S^3$ vanishes. The other terms give the solution

$$g = \tilde{B}_2\, S^3 + \tilde{B}\, \tilde{N}\, (S^3)^3 + \tilde{P}\, (S^3)^5 \tag{V-76}$$

where for simplicity we have defined the coefficients N and P by

$$\tilde{N} = -\frac{N}{\tilde{B}B(\lambda+2\mu)} \qquad \tilde{P} = -\frac{P}{24(\lambda+2\mu)} \tag{V-77}$$

and $\tilde{B}_2$ is an undetermined constant.

From the above definitions and from expressions (V-73) and (V-74) one notices that $\tilde{N}$ and $\tilde{P}$ are functions of the 5 basic parameters μ, λ, A, B, C because $\tilde{C}$ which is given by (V-69) is a function of the 4 basic parameters μ, λ, A, B. Combining both the first order and second order corrections one obtains from (V-62)

$$r_o = s^3 - \phi_2^2(\tilde{B}s^3 + \tilde{C}(s^3)^3) - \phi_2^4(\tilde{B}_2(s^3) + \tilde{B}\tilde{N}(s^3)^3 + \tilde{P}(s^3)^5) \,. \tag{V-78}$$

The two undetermined constants in the above equation are $\tilde{B}$ and $\tilde{B}_2$. To determine these coefficients one uses the fact that the lateral traction is zero or equation (V-25) where π^{33} is the term in the bracket on page 55 of equation (V-58). Substituting (V-76) and (V-70) in equation (V-25) one obtains

$$\tilde{B} = \frac{-2B(s_o^3)^2 - 6\tilde{C}(s_o^3)^2\mu - \frac{\lambda}{2}(8\tilde{C}+1)(s_o^3)^2}{2\mu + 2\lambda} \tag{V-79}$$

and

$$\tilde{B}_2 = -\frac{1}{2\mu+2\lambda}\Big\{\mu\big[(2\tilde{B}+6\tilde{C}(s_o^3)^2)^2 + 3(\tilde{B}+3\tilde{C}(s_o^3)^2)^2 + 6\tilde{B}\tilde{N}(s_o^3)^2$$

$$+10\tilde{P}(s_o^3)^4\big] + \frac{\lambda}{2}\big[8\tilde{B}\tilde{N}(s_o^3)^2 + 11\tilde{P}(s_o^3)^4 + 3(\tilde{B} + \tilde{C}(s_o^3)^2)^2$$

$$+3(\tilde{B}+3\tilde{C}(s_o^3)^2)^2 + 2(\tilde{B}+3\tilde{C}(s_o^3)^2)(4\tilde{B}+(8\tilde{C}+1)(s_o^3)^2)\big]$$

$$
\begin{aligned}
&+ B\Big[2\big(2\tilde{B} + (2\tilde{C}+1)(S_0^3)^2\big) + 8\big(\tilde{B} + 3\tilde{C}(S_0^3)^2\big)^2 \\
&+ 2\big(\tilde{B} + 3\tilde{C}(S_0^3)^2\big)\big(4\tilde{B} + (8\tilde{C}+1)(S_0^3)^2\big) \\
&\quad + 8\big(\tilde{B} + 3\tilde{C}(S_0^3)^2(S_0^3)^2\Big] + C\Big[4\tilde{B} + (8\tilde{C}+1)(S_0^3)^2\Big]^2\Big\} .
\end{aligned}
\tag{V-80}
$$

In principle one can calculate all the coefficients in terms of the basic parameters μ, λ, A, B, and C. $\tilde{C}$ is given by (V-69), $\tilde{B}$ by (V-79), then $\tilde{N}$ and $\tilde{P}$ can be calculated by (V-73) and (V-74) and finally $\tilde{B}_2$ by (V-80).

One can also find the inverse relation to equation (V-78) by finding S^3 in terms of r_0 or

$$S^3 = r_0 + \phi_2^2 \tilde{f} + \phi_2^4 \tilde{g} . \tag{V-81}$$

To find $\tilde{f}$ and $\tilde{g}$ one substitutes equation (V-81) into equation (V-78). Do an expansion in ϕ_2^2, keeping again terms up to ϕ_2^4, one obtains

$$
\begin{aligned}
S^3 &= r_0 + \phi_2^2(\tilde{B} r_0 + \tilde{C} r_0^3) \\
&+ \phi_2^4\Big\{(\tilde{B}_2 + \tilde{B}^2) r_0 + (\tilde{B}\tilde{N} + \tilde{B}\tilde{C} + 3\tilde{B}) r_0^3 + (\tilde{P} + 3\tilde{C}) r_0^5\Big\} .
\end{aligned}
\tag{V-82}
$$

From the above equation if one substitutes for r_0 the value R one gets the value for the new deformed radius S_0^3 .

Once the relationship between r_0 and S^3 is known (equation (V-78)) one can calculate the applied torque and the total applied force at the end surface.

The applied torque at the end surface is given by equation (V-30) or

$$M = \int_0^{S_0^3} \pi^{12} (S^3)^3 \, dS^3 \tag{V-83}$$

where π^{12} is given by the 1,2 element of π given by equation (V-55). Hence,

$$\begin{aligned} M = 2\pi \int_0^{S_0^3} \Bigg\{ & \mu\left(\phi_2\left(\frac{S^3}{r_0}\right)^2 + \phi_2^3 (S^3)^2\right) + \frac{\lambda}{2}\left(\left(\frac{S^3}{r_0}\right)^2 - 2 + \phi_2^2 (S^3)^2 + \frac{1}{\left(\frac{\partial r_0}{\partial S^3}\right)^2}\right)\phi_2 \\ & + A\left(\left(\left(\frac{S^3}{r_0}\right)^2 - 1 + \phi_2^2 (S^3)^2\right)^2 \phi_2 + \left(\left(\frac{S^3}{r_0}\right)^2 - 1 + \phi_2^2 (S^3)^2\right)\phi_2\right) \\ & + 2B\phi_2 \left(\left(\left(\frac{S^3}{r_0}\right)^2 - 1 + \phi_2^2 (S^3)^2\right)^2 + 2\phi_2^2 (S^3)^2 + \left(\frac{1}{\left(\frac{\partial r_0}{\partial S^3}\right)^2} - 1\right)^2\right) \\ & + B\left(\left(\frac{S^3}{r_0}\right)^2 - 2 + \phi_2^2 (S^3)^2 + \frac{1}{\left(\frac{\partial r_0}{\partial S^3}\right)^2}\right)\left(\phi_2\left(\frac{S^3}{r_0}\right)^2 + \phi_2^3 (S^3)^2\right) \\ & + C\left(\left(\frac{S^3}{r_0}\right)^2 - 2 + \phi_2^2 (S^3)^2 + \frac{1}{\left(\frac{\partial r_0}{\partial S^3}\right)^2}\right)^2 \Bigg\} (S^3)^3 \, dS^3. \end{aligned} \tag{V-84}$$

Doing the S^3 integration and using the result (V-78) giving r_0 as a function of S^3 one obtains , if one keeps terms up to order ϕ_2^3, the following results:

$$\begin{aligned} M = \frac{\pi}{2}\phi_2 \mu (S_0^3)^4 + \phi_2^3 \Big\{ & \frac{\pi}{2}(S_0^3)^4 (2\mu + 2\lambda + 2A + 4B)\tilde{B} \\ & + \frac{\pi}{3}(S_0^3)^6 \left(\tilde{C}(2\mu + 4\lambda + 2A + 8B) + \mu + \frac{\lambda}{2} + 4A + 5B\right\} . \end{aligned} \tag{V-85}$$

If one uses (V-82) because $S_0^3 = S^3$ ($r_0 = R$) again keeping terms up to order ϕ_2^3 one obtains

$$M = \frac{\pi}{2}\phi_2 \mu R^4 + \phi_2^3 \pi R^4 \Big\{ (2\mu + 2\lambda + 2A + 4B)\frac{\widetilde{B}}{2}$$

$$+ 2\mu(\widetilde{B} + \widetilde{C}(R^2)) + \frac{\widetilde{C}}{3}R^2(2\mu + 4\lambda + 2A + 8B) + \frac{R^2}{3}(\mu + \frac{\lambda}{2} + 4A + 5B)\Big\} . \qquad \text{(V-86)}$$

Expression (V-86) in the limit of small torsion ($\phi_2 \to 0$) gives the result of classical elasticity for small displacements, that is to say $\frac{\pi}{2}\phi_2 \mu R^4$. For larger displacement one must include a term in ϕ_2^4 .

In the above expression $\widetilde{B}$, $\widetilde{C}$ are constants given by (V-69) and (V-79) and M can be calculated as a function of R and the constants μ, λ, A, B. One notices that the constant C does not enter into the expression (V-86).It would enter in the expression for M if one carried the calculation up to order ϕ_2^4 .

The z component of the traction at the end of the cylinder is given by (V-29) or with the help of equation (V-55) by

$$f_z = \mu\phi_2^2(s^3)^2 + \frac{\lambda}{2}\left(\left(\frac{s^3}{r_0}\right)^2 - 2 + \phi_2^2(s^3)^2 + \frac{1}{\left(\frac{\partial r_0}{\partial s^3}\right)^2}\right) + A\left\{\left(\left(\frac{s^3}{r_0}\right)^2 - 1 + \phi_2^2(s^3)^2\right)\phi_2^2(s^3)^2\right.$$

$$\left. + \phi_2^2(s^3)^2\right\} + 2B\left\{\left(\left(\frac{s^3}{r_0}\right)^2 - 1 + \phi_2^2(s^3)^2\right)^2 + 2\phi_2^2(s^3)^2 + \left(\frac{1}{\left(\frac{\partial r_0}{\partial s^3}\right)^2} - 1\right)^2\right\}$$

$$+ B\left\{\left(\frac{s^3}{r_0}\right)^2 - 2 + \phi_2^2(s^3)^2 + \frac{1}{\left(\frac{\partial r_0}{\partial s^3}\right)^2}\right\}\phi_2^2(s^3)^2 + C\left\{\left(\frac{s^3}{r_0}\right)^2 - 2 + \phi_2^2(s^3)^2 + \frac{1}{\left(\frac{\partial r_0}{\partial s^3}\right)^2}\right\}^2 . \qquad \text{(V-87)}$$

Doing an expansion in terms of ϕ_2 and keeping terms to order ϕ_2^2 only one gets

$$f_z = \phi_2^2\left\{\mu(s^3)^2 + \frac{\lambda}{2}(4\widetilde{B} + (8\widetilde{C} + 1)(s^3)^2) + (A + 4B)(s^3)^2\right\} . \qquad \text{(V-88)}$$

As in the theory of rubber like solids the traction is function of the square of the torsion angle. It is either positive or negative, depending on the values of the elastic moduli.

From the above equation for the traction one can calculate the total force in the z direction at the end of the cylinder. It is given by equation (V-31) or

$$F = \phi_2^2 \left\{ \mu + \frac{\lambda}{2}\left(8\tilde{C}+1\right) + A + 4B \right\} \frac{\pi}{2}(S_o^3)^4 + \phi_2^2 2\lambda\pi(S_o^3)^2\tilde{B} \qquad \text{(V-89)}$$

or making use of (V-82)

$$F = \phi_2^2 \left((\mu + \frac{\lambda}{2}(8\tilde{C}+1) + A + 4B) \frac{\pi}{2} R^4 + 2\pi\lambda\tilde{B} R^2 \right) . \qquad \text{(V-90)}$$

Another approach for calculating the new radius, the torque at the end of the cylinder and the force at the end of the cylinder is to do a numerical computation with the help of a computer. This will be a first order calculation in the torsion parameter.

To do this one first calculates the new radius.One has to solve simultaneously the equation for the radius, equation (V-78) and the equation for the constant, $\tilde{B}$, in terms of the radius, equation (V-79).

It is better to render all the equations dimensionless. One will prime all the dimensionless quantities

$$S_o^{3\prime} = \frac{S_o^3}{R} \qquad \text{(V-91)} \qquad\qquad \tilde{B}' = \frac{\tilde{B}}{R^2} \qquad \text{(V-92)}$$

$$Q1 = \frac{\lambda}{\mu}, \quad Q2 = \frac{A}{\mu}, \quad Q3 = \frac{B}{\mu}, \quad \text{and} \quad Q4 = \frac{C}{\mu}. \tag{V-93}$$

One can introduce a dimensionless torsion angle

$$\phi_2' = \phi_2 R . \tag{V-94}$$

With the help of equations (V-91), (V-92), (V-93), (V-94) equations (V-78), (V-79) and (V-69) reduce to

$$\tilde{C} = -\frac{1}{8}\left(\frac{Q1 - 2 + 4Q3 - Q2}{Q1 + 2}\right), \tag{V-95}$$

$$\tilde{B}' = \left(\frac{-2Q3 - 6\tilde{C} - \frac{Q1}{2}(8\tilde{C} + 1)}{2 + 2Q1}\right)(S_0^{3}{}')^2 \tag{V-96}$$

and

$$S_0^{3}{}' = 1 + \phi_2^{2}{}'(\tilde{B}' S_0^{3}{}' + \tilde{C}(S_0^{3}{}')^3) . \tag{V-97}$$

One calculates $S_0^{3}{}'$ and $\tilde{B}'$ by simple iteration with the help of a program in Fortran V called E5 . One calculates the constant $\tilde{C}$ for a set of parameters Q1, Q2, Q3, Q4 and then by setting $S_0^{3}{}'$ = 1 one calculates $\tilde{B}'$ and $S_0^{3}{}'$ by equations (V-96) and (V-97). One then uses the value obtained for $S_0^{3}{}'$ and substitutes into equation (V-96) to calculate a new value for $\tilde{B}'$. Substituting into the right hand side of equation (V-97) to obtain a new value of $S_0^{3}{}'$, one then repeats the same procedure.

After a few iterations the procedure converges to the solution

, $S_0^{3'}$, of the simultaneous equations. The numbers converge to within 10^{-5}.

The above procedure is carried out for different values of ϕ_2' between 0 and a maximum value that can be varied. As one will see in the discussion this maximum value is set at 0.06. One then plots the values of $S_0^{3'}$ versus ϕ_2' using a subroutine obtained from the computer library.

To calculate the torque one also uses the dimensionless quantity M',

$$M' = \frac{M}{M_0} \quad . \tag{V-98}$$

M_0 is the torque at the end of the cylinder obtained from the classical theory of elasticity

$$M_0 = \mu \phi_2 \frac{\pi}{2} R^4 \quad . \tag{V-99}$$

To calculate the dimensionless torque M' one uses equation (V-84) for the torque and substitutes in this equation the equations for the radius to first order, equation (V-78). What is meant by first order is really a term in ϕ_2^2 since there are no terms in ϕ_2 .

After some algebra and reduction to dimensionless variables one gets

$$M' = M'_\mu + M'_\lambda + M'_A + M'_B + M'_{2B_1} + M'_{2B_2} + M'_C \; . \tag{V-100}$$

with

$$M'_\mu = (S_o^{3\prime})^4 \{ 1 + 2\phi_2'^2 \tilde{B}' + (\tfrac{4}{3}\tilde{C} + \tfrac{2}{3}) \phi_2'^2 (S_o^{3\prime})^2 \} , \qquad (V\text{-}101)$$

$$M'_\lambda = 2Q1\, \phi_2'^2 \{ \tilde{B}'(S_o^{3\prime})^4 + (8\tilde{C}+1) \frac{(S_o^{3\prime})^6}{6} \} , \qquad (V\text{-}102)$$

$$M'_A = 4Q2\, \phi_2'^2 (S_o^{3\prime})^4 \{ \frac{\tilde{B}'}{2} + (2\tilde{C}+2) \frac{(S_o^{3\prime})^2}{6} \} + 4Q2\, \phi_2'^4 (S_o^{3\prime})^4$$

$$\{ \tilde{B}'^2 + \frac{2}{3} \tilde{B}'(2\tilde{C}+1)(S_o^{3\prime})^2 + (2\tilde{C}+1)^2 \frac{(S_o^{3\prime})^4}{8} \} , \qquad (V\text{-}103)$$

$$M'_B = 4Q3 (S_o^{3\prime})^4 \{ \phi_2'^2 \frac{(S_o^{3\prime})^2}{3} + \phi_2'^4 (2\tilde{B}'^2 + (40\tilde{C}^2 + 4\tilde{C}+1) \frac{(S_o^{3\prime})^4}{8}$$

$$+ \frac{2}{3} \tilde{B}'(8\tilde{C}+1)(S_o^{3\prime})^2 \} , \qquad (V\text{-}104)$$

$$M'_{2B_1} = 8\,Q3\, \phi_2'^2 (S_o^{3\prime})^4 (\tilde{B} + (8\tilde{C}+1) \frac{(S_o^{3\prime})^2}{6}) , \qquad (V\text{-}105)$$

$$M'_{2B_2} = 8Q3\, \phi_2'^4 (S_o^{3\prime})^4 (2\tilde{B}'^2 + \tilde{B}'(4\tilde{C}+1)(S_o^{3\prime})^2 + (8\tilde{C}+1)(2\tilde{C}+1) \frac{(S_o^{3\prime})^4}{8}) , \qquad (V\text{-}106)$$

$$M'_c = 4Q4\, \phi_2'^4 (S_o^{3\prime})^4 (4\tilde{B}'^2 + \frac{4}{3} \tilde{B}'(8\tilde{C}+1)(S_o^{3\prime})^2 + (8\tilde{C}+1)^2 \frac{(S_o^{3\prime})^4}{8}) . \qquad (V\text{-}107)$$

With the results of the previous computer calculation for $S_o^{3\prime}$ one can then calculate the dimensionless torque as a function of ϕ_2' and use a library subroutine to plot the result.

To calculate the force F at the end of the cylinder one introduces the dimensionless variable

$$F' = \frac{F}{F_o} \tag{V-108}$$

where F_o is the force which when applied on the side of the cylinder, would give a torque M_o,

$$F_o = \frac{M_o}{R} \quad . \tag{V-109}$$

The force F' is calculated from the expression (V-87) for the traction at the end of the cylinder and equation (V-31) for the force F. If one also uses the dimensionless quantity F' given by (V-108), one obtains

$$F' = F_\mu' + F_\lambda' + F_A' + F_B' + F_{2B}' + F_C' \tag{V-110}$$

with the following forces

$$F_\mu' = \phi_2' (S_o^{3\prime})^4 \quad , \tag{V-111}$$

$$F_\lambda' = \left(\frac{\varphi 1}{2}\right) \phi_2' (S_o^{3\prime})^2 \left(4\tilde{B}' + (8\tilde{C}+1)\left(\frac{S_o^{3\prime}}{2}\right)^2 \right) \quad , \tag{V-112}$$

$$F_A' = (Q2)\,\phi_2'(S_o^{3\prime})^4\left(1 + \phi_2'^2\left(2\tilde{B}' + (2\tilde{C}+1)(S_o^{3\prime})^2\,\tfrac{2}{3}\right)\right) \;, \qquad \text{(V-113)}$$

$$F_B' = \left(\frac{Q3}{2}\right)\phi_2'(S_o^{3\prime})^2\Big\{\phi_2'^2\big(32\tilde{B}'^2 + 8\tilde{B}'(8\tilde{C}+1)(S_o^{3\prime})^2 + \tfrac{4}{3}(40\tilde{C}^2 + 4\tilde{C}+1)(S_o^{3\prime})^4\big) + 4(S_o^{3\prime})^2\Big\} \;, \qquad \text{(V-114)}$$

$$F_{2B}' = 2(Q3)\,\phi_2'^3(S_o^{3\prime})^4\left\{4\tilde{B}' + \tfrac{2}{3}(8\tilde{C}+1)(S_o^{3\prime})^2\right\} \;, \qquad \text{(V-115)}$$

and

$$F_C' = (Q4)\,\phi_2'^3\left\{32\tilde{B}'^2(S_o^{3\prime})^2 + 8\tilde{B}'(8\tilde{C}+1)(S_o^{3\prime})^4 + \tfrac{2}{3}(8\tilde{C}+1)^2(S_o^{3\prime})^6\right\}. \qquad \text{(V-116)}$$

Again one uses the results of the previous computer calculation for $S_0^{3\prime}$ which when substituted into (V-110) and (V-111) - (V-116) will give the force at the end of the cylinder. The calculation is repeated for different values of ϕ_2' , and the result plotted with the help of a computer library subroutine.

CHAPTER VI

Discussion and Conclusion

In the previous section one calculated the radius, torque and force at the end of the cylinder as a function of the dimensionless torsion angle ϕ_2' . The results of the computations depend on the two Lamé coefficients μ and λ and the three third order elastic constants or elastic moduli A,B, and C. These elastic moduli are sometimes called second order elastic constants since they are the next terms following the Lamé coefficient terms.

A microscopic description of the nature of the third order elastic constant is rather difficult since one would have to know exactly the interatomic forces as a function of the separation and angular displacement of the atoms forming the substance. For a metallic compound these forces are the ion-ion interactions screened by the conduction electrons. For nonmetallic compounds these forces are of molecular origin.

Looking at equations (IV-20) and (IV-21) one notices that all the constants A,B,C contribute to the isotropic compression of a material but only the constant A contributes to a simple shear of a material. Hence, A can be considered as a higher order shear modulus.

Third order elastic constants can be found in the literature. They are often called ν_1, ν_2, ν_3, for isotropic materials[18], l,m,n [9] or A',B',C' [19]. A relation between these different definitions of the third order elastic constants can be found in R.T. Smith, R.Stern, R.W. Stevens [13].

The third order elastic constant used in this work, A, B, C are one quarter of the constants A',B',C'. There are two reasons for this. The first one being that one avoids factors of 1/2 in the definitions of the strain tensor. The second is that our constants A, B, C are of the same order of magnitude as the Lamé coefficients μ and λ instead of having large values (Table 2).

Third order elastic constants are obtained from the measurement of the ultrasonic velocity. Two main methods are used one involving the variation of pulse transit time versus an applied stress the other by measurement of the spacial damping of longitudinal and transverse ultrasonic waves.

In the first method one obtains the so called mixed constants related to the temperature derivatives of adiabatic constants. In the second method one obtains the adiabatic constants. In the data given in the literature no distinction is made between adiabatic and mixed constants. In this work one is interested in isothermal constants. In Appendix I the relations between the adiabatic and isothermal elastic constants are derived. An estimate of the difference between the second order adiabatic and isothermal elastic constant λ shows that the difference is less than 1 percent. For the third order elastic constants the constant A is the same for adiabatic and isothermal cases. The difference between the constants B and C is even smaller, and is not needed since the accuracy of the measurement of the third order elastic constants is very low [Smith, Sterns, Stevens [13]].

Most of the measurements of the third order elastic constants are made for single crystals, but some have been made for polycrystalline materials. One can find in the literature experimental results for six metallic compounds and three non metallic ones. Although the ultrasonic measurement techniques are quite accurate (few parts in 10^6) the results for the third order elastic constants which are derived from the experimentals results are less so. They are accurate to a few tens of percent. The percent error for C' is even higher since it is usually obtained from the difference of two approximatly equal numbers. The percentage errors vary from low values for magnesium plates (0.9, 1.7, 1.5) for A',B',C' respectively to the high values for molybdenum (2, 10, 100), respectively. The values for aluminum alloys are typically accurate to 5,10 and 20 percent, respectively. One should point out that one has not included the recent data of Raju and Reddy (17) on aluminum-magnesium alloys because of their large percentage errors since the data for other aluminum alloys which have much smaller percentage errors have been included.

The third order elastic constants have mostly negative values for the metallic compounds and the constants A,B,C (one quarter of the constants A',B',C') have the same order of magnitude as the constants μ and λ .Only the helca steel and the resintered molybdenum have positive constants C'. Since the relative errors in measuring C' are large for these two values, these two results are rather uncertain. There is also a positive value for A' for columbium(15) which is accurate to 7 percent.

For non-metallic compounds the data for pyrex [14] has very large percentage errors (85 for A', 200 for B', 200 for C'). The data for polysterene has percentage errors of 14 for A', 43 for B',and 64 for C'. The percentage errors for the third order elastic constants for fused silica are small (2.2 for A', 6 for B', 6 for C').

In Table 2 one has tabulated the ratio of the elastic constants λ ,A,B,C to μ . One calls these ratios Q1,Q2,Q3,Q4 (equation (V-93)).

One has chosen one typical aluminium alloy and one typical steel. Some interesting results occur. Q1, the ratio of $\frac{\lambda}{\mu}$, is almost equal for sintered and resintered samples of the same material although there is almost of factor of two between the values of μ and λ for these samples.

Most of the ratio Q2,Q3,Q4 of the third order elastic constants to μ are of the order of unity. The values of Q2 are larger with a maximum ratio of 4 for Columbium.

To do numerical calculations one would like to get some idea of the order of magnitude of the dimensionless torsion angle ϕ_2 R . To do this one considers the deformation of a small cube of side Δa during the twisting of the cylindrical rod. As can be seen in figure (7) a small cube gets distorted. The largest distorsion will occur for a small cube whose upper surface coincides with the lateral surface of the cylinder. One can calculate the maximum relative elongation for the twisting. Since the change in radii are of second order in the torsion parameter one can neglect them and calculate the relative change in the length AC which after the twisting becomes A'C',figure (7).

The original length AC is $\sqrt{2}\,\Delta a$. After torsion, looking at the triangle A'C'E' one obtains

$$A'C' = \sqrt{(\Delta a)^2 + (\Delta a + \phi_2' \Delta a)^2} \qquad \text{(VI-1)}$$

since the length E'B' is given by

$$E'B' = (\phi_2 \Delta a) R = \phi_2' \Delta a \quad .$$

From equation (VI-1), after an expansion to first order in one easily obtains

$$\phi_2' = 2\left(\frac{A'C' - AC}{AC}\right) \quad . \qquad \text{(VI-3)}$$

Hence, from equation (VI-3) one concludes that the dimensionless torsion angle is of the same magnitude as twice the relative elongation.

One has chosen to run the calculations up to a value of ϕ_2' equal to 0.06 even though this might correspond to a regime where plastic deformations already set in. However it is not very clear for some materials at what strain value the plastic deformations begin.

One can look at the results for the radius, torque and force seperately.

The radius is in dimensionless form $\frac{S_o^3}{R} = S_o^{3\prime}$ and is given as a function of the dimensionless torsion angle ϕ_2'. The radius decreases when the third order elastic constants are set to zero.

One can call this the linear regime, but one has to keep in mind the deformations are large, and the results are not the same as the one of the classical theory of elasticity for small displacements. The change of radius is small and decreases as the ratio of increases. It goes from 0.06 percent to 0.003 percent. A typical value for aluminium is 0.06 percent.

In the computer printout one obtains the dimensionless radius $S_0^{3'}$ as a function of ϕ_2' . To compare the results of the different materials one can choose the value of ϕ_2' to be 0.06. If need be, one can look at the computer data and compare the values for other ϕ_2' .

When the third order elastic constants are added the radius now increases for most metals with the exception of Columbium. This increase varies from a low of 0.013 percent for resintered tungsten to a maximum of 0.1 percent for aluminium. For Columbium the decrease is small, 0.0002 percent, and hence, the radius can be considered almost constant.

For non-metals the radius decreases for pyrex by 0.06 percent and also decreases for fused silica by 0.1 percent but increases for polysterene by 0.05 percent.

One should note that when the torsion angle parameter is smaller, the change of radii are much smaller. For example in aluminium when ϕ_2' is 0.02 the relative increase of radius is only 0.011 percent.

Although the experiment performed by Poynting[26] does not correspond to the situation examined in this work, one can use the experimental results of Poynting and his theory to calculate the change of radius of a steel rod whose ends are prevented from moving. One obtains an increase in radius roughly one third smaller than the one calculated in this work. This discrepancy is probably due to the use of the phenomological theory of Poynting which does not reduce to the results of Murnaghan[9] or to the results of this work.

Another problem with Poynting's experiment is that the wires he used were subjected to large electrical currents to straighten them. This created a permanant circularly magnetised wire and also oxydised the outer surface of the wire which was then robbed out.

The dimensionless torque at the end of the cylinder is plotted by program E5 as a function of the dimensionless torsion angle ϕ_2' . If the classical theory of elasticity is valid the dimensionless torque M' would be constant and equal to one. This is almost the case, especially when the third order elastic constants are set to zero, linear regime. In this regime the dimensionless torque M' decreases as the dimensionless torsion angle increases. The deviation from one varying from - 0.2 percent for tungsten to - 0.07 percent for Columbium. When the third order elastic constants are included the deviations from one vary from - 0.04 percent for resintered tungsten to - 2 percent for aluminium.

One has also calculated and plotted the torque at the end of the cylinder M measured in dynes-cm versus the dimensionless parameter ϕ_2', Program E5'. The torque has been calculated for an initial radius R of 1cm. The torque varies almost linearly as a function of ϕ_2' as one would expect from the previous discussion. The range of values for the torque vary from 0.156 x 10" dynes-cm for magnesium to a value of 1.29 x 10" dynes-cm for resintered tungsten.

For the non-metallic compounds the deviation from one are - 3 percent for pyrex, - 0.4 percent for fused silica and - 1 percent for polysterene. The torque M for pyrex is .251 x 10" dynes-cm, .293 x 10" dynes-cm for fused silica and 0.0128 x 10" dynes-cm for polysterene.

The force at the end of the cylinder which keeps the cylinder from expanding or contracting is plotted in its dimensionless form F' versus the dimensionless torsion angle ϕ_2' by program E5. In the elastic regime this force is close to a linear function and is positive. For the metallic compounds it decreases as the ratio of $\frac{\lambda}{\mu}$ increases from a value of 0.045 for tungsten to a value of 0.039 for Columbium.

When the third order elastic constants are included the dimensionless force at the end of the cylinder becomes negative for metallic compounds. It varies from - 0.061 for sintered tungsten to -0.182 for aluminium.

The force F at the end of the cylinder is plotted by program E5' as a function of the dimensionless torsion parameter ϕ_2' also for a cylinder of initial radius 1cm. This force varies from a value of - 0.016 x 10" dynes for magnesium to a value of - 0.0875 x 10" dynes for resintered tungsten.

For non-metallic compounds the dimensionless force for pyrex is + 0.154, + 0.063 for fused silica and - 0.093 for polysterene. The force F at the end of each cylinder are + 0.0399 x 10" dynes for pyrex, + 0.0186 x 10" dynes for fused silica and -0.00121 x 10" dynes for polysterene.

One can summarize the results of this work as follows.

The first three chapters are general and describe the body tensor formalism and the physical laws described in this formalism. The description of strain is based on the definition of the body metric tensors at time t and t_0 given by equations (II-4) and (II-5). One can either deal with the covariant strain tensor $\gamma-\gamma_0$ or with a mixed strain tensor given by equation (III-23). The description of stress is similar to the description of stress in a space coordinate system and the stress tensor π like the stress tensor in a space coordinate system is also symmetric, equation (II-34). Although one examined the physical laws for a moving system, if one is only interested in elastostatic problems, the condition of equilibrium is the divergence of the stress tensor being zero, equation (III-12). Since one usually deals with a curvilinear body coordinate system the divergence of the stress tensor can be evaluated in terms of the Christoffel symbols by equation (III-10). The Christoffel symbols can then be evaluated from the body metric tensor and its derivatives, equations (II-28) and (II-29).

Furthermore, if one deals with isothermal processes the constitutive equation relating the stress to the strain is equation (III-27) which relates the stress to the derivative with respect to the body metric tensor of the free energy of the strained body. The free energy of the strained body is the part of the free energy independent of temperature, equation (III-22).

A useful approach to a constitutive equation is to expand the free energy in terms of strains, keeping quadratic and cubic terms. One then obtains the constitutive equation (IV-21). It involves the two Lame coefficients and three third order elastic constants A, B, and C. This equation although obtained from the free energy could be considered an ad-hoc generalization of Hook's Law. One has applied the body tensor formalism to the torsion of a right circular cylinder for which one has to solve a non-linear differential equation, (V-58). One has solved this equation to first order in the torsion parameter and calculated the new radius, the torque and the force at the end of the cylinder. The result for the new radius is the same as the result obtained by Murnaghan [9]. The new radius being a function of the two Lamé coefficients and the two third order elastic moduli A and B.

The result for the torque at the end of the cylinder is different.The torque at the end of the cylinder depends on the two Lame coefficients and the third order elastic moduli A, B, and C.

The force at the end of the cylinder also depends on the two Lame coefficients and the three elastic modulii A, B, C. It is either positive or negative (traction or compression) depending on the relative values of the elastic moduli.

If one assumes a rubber like solid, since the material is incompressible, there is no change in radius. The expression for the torque at the end of the cylinder is almost the same as for an elastic solid but the force at the end of the cylinder is a compression. Hence, it is qualitatively different from the force of the elastic solid as seen in the paragraph above.

In this work the elastic moduli were obtained from the literature on ultrasonics, and the above quantities were calculated as a function of the torsion parameter.

Another interesting approach would be to measure the new radius, the torque and the force as a function of the torsion angle. With the knowledge of the two Lame coefficients and the result of the calculations one could obtain the three third order elastic constants A, B and C.

One could extend the body tensor formalism to the consideration of plastic deformations, but the task is not easy since the separation of the elastic and plastic parts in a deformation involves the introduction of an "unloading state" which might be non-physical, that is to say representing a non-Euclidian state, (A. V. Skachenko - A. N. Sproykhin [20]).

A possible extension of the finite deformation elasticity to the plastic domain might involve the introduction of stress or strain dependant elastic moduli.

Other problems where the introduction of body tensors would be interesting are problems of thermo-elasticity but the added complexity of irreversible thermodynamics makes this a difficult problem . Freed[5].

APPENDIX I

Relationship Between Isothermal and Adiabatic Elastic Constants

To investigate the relationship between isothermal and adiabatic elastic constants one can without much loss of generality consider the infinitesimal deformations of an elastic solid in an orthogonal space coordinate system. In this coordinate system covariant, contravariant and mixed strain tensors are the same. Hence, one can use covariant components for the stresses and strains which one will call σ_{ij} and u_{ij}.

The free energy per unit volume is given by equation (III-22) or

$$\tilde{F}(u_{ij},T) = F_0(T) + [C(u_{ij})](T-T_0) + F(u_{ij}) \tag{AI-1}$$

where in the above equation one keeps only linear terms in the temperature difference $T-T_0$ since one assumes that there are only small changes in temperature. The free energy of the strained body is given by the constitutive equation (IV-20)

$$F(u_{ij}) = \mu\, u_{ik}u_{ki} + \frac{\lambda}{2}u_{ii}^2 + \frac{A}{3}u_{ik}u_{kj}u_{ji} + B\, u_{ik}u_{ki}u_{jj} + \frac{C}{3}u_{ii}^3 . \tag{AI-2}$$

The entropy per unit volume can be obtained from the free energy by taking the derivative of the free energy with respect to temperature at constant strain

$$S = -\left.\frac{\partial\tilde{F}}{\partial T}\right)_{u_{ij}} . \tag{AI-2'}$$

Making use of the above equation and the expression for the free energy equation (AI-1) one obtains

$$S = S_0 (T) - C (u_{ij}). \tag{AI-3}$$

Hence, to first order in the temperature changes the entropy is separable into a part representing the entropy of a strain free state and a part due to the strain contributions.

If one considers an adiabatic process, the entropy stays constant; therefore,

$$S(T) = S_0(T_0) \tag{AI-4}$$

where in the above equation T_0 is the initial temperature of the medium which is assumed to be in thermodynamic equilibrium.

To calculate the change in temperature during an adiabatic process one can use equation (AI-4) and expand the left hand side of the equation in a Taylor's series keeping only first order terms in the temperature change. One obtains

$$S_0(T_0) + \left.\frac{\partial S_0}{\partial T}\right)_V (T-T_0) - C(u_{ij}) = S_0(T_0) \ . \tag{AI-5}$$

Making use of the themodynamic expression

$$\left.\frac{\partial S_0}{\partial T}\right)_V = \frac{C_V}{T_0} \tag{AI-6}$$

where in the above expression C_v is the specific heat at constant volume per unit volume one obtains

$$T - T_0 = \frac{C(u_{ij}) T_0}{C_v} . \tag{AI-7}$$

The stress can be obtained by the strain derivative of the free energy, equation (III-27) or

$$\sigma_{ij} = \left.\frac{\partial \tilde{F}}{\partial u_{ij}}\right)_T . \tag{AI-8}$$

With the help of the above equation, and equations (AI-1) and (AI-2) one obtains

$$\begin{aligned} \sigma_{ij} = {} & \lambda u_{ll} \delta_{ij} + 2\mu u_{ij} + A u_{il} u_{lj} + 2B u_{il} u_{mm} \\ & + B u_{mn}^2 \delta_{ij} + C u_{rr}^2 \delta_{ij} + \frac{\partial C(u_{ij})}{\partial u_{ij}} (T - T_0) . \end{aligned} \tag{AI-9}$$

The adiabatic elastic constants which one can call μ^A, λ^A, A^A, B^A, C^A can be defined by requiring the stress strain relationship to have the same form for adiabatic or isothermal processes; hence,

$$\begin{aligned} \sigma_{ij} = {} & \lambda^A u_{ll} \delta_{ij} + 2\mu^A u_{ij} + A^A u_{il} u_{lj} + 2B^A u_{ij} u_{mm} \\ & + B^A u_{mn}^2 \delta_{ij} + C^A u_{rr}^2 \delta_{ij} . \end{aligned} \tag{AI-10}$$

To compare equation (AI-10) to equation (AI-9) with the temperature change given by the equation (AI-7) one needs to know the function $C(u_{ij})$. Since one is interested only in second order terms,it is sufficient to write the most general scalar function of second order

$$C(u_{ij}) = \alpha_1 u_{mm} + \frac{\alpha_{2\lambda}}{2} u_{mm}^2 + \alpha_{2\mu} u_{mn}^2 \qquad \text{(AI-11)}$$

where α_1, $\alpha_{2\lambda}$, $\alpha_{2\mu}$ are constants. As can be seen later α_1 is related to the coefficient of linear expansion; hence, $\alpha_{2\lambda}$ and $\alpha_{2\mu}$ can be considered second order coefficients of linear expansion. The μ and λ indices are chosen to remind one of the similarity of the last two terms of the right hand side of equation (AI-11) and the first two terms of the right hand side of equation (AI-2).

To find the adiabatic elastic constants one uses equation (AI-11) to calculate the temperature change from equation (AI-7), calculates $\frac{\partial C(u_{ij})}{\partial u_{ij}}$ from equation (AI-11) and substitutes all the results into equation (AI-9). By comparing this equation and equation (AI-10) one obtains

$$\mu^A = \mu \ , \qquad \text{(AI-12)}$$

$$\lambda^A = \lambda + \frac{\alpha_1^2 T_0}{C_v} \ , \qquad \text{(AI-12')}$$

$$A^A = A \ , \qquad \text{(AI-12'')}$$

$$B^A = B + \alpha_1 \alpha_{2\mu} \frac{T_0}{C_v} \ , \qquad \text{(AI-12''')}$$

and

$$C^A = C + \frac{3}{2} \alpha_1 \alpha_{2\lambda} \frac{T_0}{C_v} \quad . \tag{AI-12''''}$$

The above equations relate the adiabatic and isothermal elastic constants. The coefficients μ and A which give rise to shear problems are equal for the adiabatic and isothermal processes, the other coefficients are different.

One can transform the relation (AI-12') with some further definitions.

Define α_1 to be

$$\alpha_1 = -K\alpha \tag{AI-13}$$

where K is the compression modulus given by,

$$K = \lambda + \frac{2}{3} \mu \quad , \tag{AI-14}$$

and α is the coefficient of thermal expansion. It is the coefficient of volume expansion, sometimes denoted by β

Defining the adiabatic compression modulus with an equation similar to (AI-14)

$$K_A = \lambda^A + \frac{2}{3} \mu$$

and making use of equations (AI-13), (AI-14), (AI-12), (AI-12') one obtains

$$K_A = K + \frac{K^2\alpha^2 T_0}{C_v} \; . \tag{AI-15}$$

If one use the thermodynamic identity relating the specific heats,

$$C_p - C_v = K\,\alpha^2 T_0, \tag{AI-16}$$

one can transform equation (AI-15) to

$$\frac{1}{K_A} = \frac{1}{K} - \frac{T_0 \alpha^2}{C_p} \; . \tag{AI-17}$$

The above equation can be derived from general thermodynamic arguments (Landau-Lifshitz [7]).

APPENDIX II

Christoffel Symbols and Curvature Tensor

The Christoffel symbols are introduced in tensor analysis when one deals with derivatives of vectors and tensors. Hence, to consider the Christofell symbols one must examine the concept of the differential of a vector. One will follow the arguments developed by Landau and Lifshitz [(21)]. They may lack mathematical rigor but they gain in clarity due to their geometrical interpretation.

To consider the differential of a covariant vector $\vec{A}$ one must calculate the change of this vector, calculate $\vec{A}\ (S^i + dS^i) - \vec{A}(S^i)$ where one considers the vector $\vec{A}$ at a point M and the vector $\vec{A}$ at a point M' an infinitesimal distance away from the point M (fig. 9). To achieve this one must "parallel transport" the vector A from the point M to the point M'. This parallel transport is simply a translation operation. In this parallel transport since one is using curvilinear coordinates the components of the vectors change. Since the change must be a linear function of the components themselves and must be proportional also to the infinitesimal coordinate changes, one must have

$$\delta A^i = -\Gamma^i_{kl}\ A^k \delta S^l \tag{AII-1}$$

where Γ^i_{kl} is a certain function of the coordinates called the Christoffel symbol of the second kind. In fig (9) δA^i corresponds to the difference between M'N' and MN, M'N' being the S^1 component of the vector $\vec{A}'$ which is the parallel transport of the vector $\vec{A}$ and

MN being the S^1 component of the vector $\vec{A}$.

The differential of the vector $\vec{A}$ is the difference of the vector $\vec{A}$ $(S^i + dS^i)$ and the vector $\vec{A}'$ both being defined at the same point M'; therefore,

$$DA^i = dA^i - \delta A^i \qquad \text{(AII-2)}$$

where dA^i is the change of the contravariant component of the vector $\vec{A}$ due to a change of coordinate

$$dA^i = \frac{\partial A^i}{\partial S^j} dS^j . \qquad \text{(AII-3)}$$

Substituting the above equation into equation (AII-2) and then using equation (AII-1) one obtains

$$DA^i = \left(\frac{\partial A^i}{\partial S^l} + \Gamma^i_{kl} A^k \right) dS^l . \qquad \text{(AII-4)}$$

Hence, the contravariant differential of the vector $\vec{A}$ which is also a contravariant vector not only involves the differential of the components but also a term (second term in the right hand side of the above equation) which takes into account the change of the basis vectors. If one deals with a cartesian coordinate system, this term is identically zero.

Since for a parallel displacement a scalar is unchanged, one has

$$\delta (\vec{A} \cdot \vec{B}) = 0. \qquad \text{(AII-5)}$$

or

$$\delta\,(A^i B_i) = 0. \qquad \text{(AII-5')}$$

From the above equation, following a procedure similar to the derivation of equation (AII-4) one obtains

$$DA_i = \left(\frac{\partial A_i}{\partial S^l} - \Gamma^k_{il}\, A_k \right) dS^l. \qquad \text{(AII-6)}$$

One can also define the covariant derivatives by

$$DA^i = A^i{}_{,l}\; dS^l \quad ; \quad DA_i = A_{i,l}\; dS^l. \qquad \text{(AII-7)}$$

Hence, these covariant derivatives are given by the following equations

$$A^i{}_{,l} = \frac{\partial A^i}{\partial S^l} + \Gamma^i_{kl}\, A^k \qquad \text{(AII-8)}$$

and

$$A_{i,l} = \frac{\partial A_i}{\partial S^l} - \Gamma^k_{il}\, A_k \qquad \text{(AII-9)}$$

Equations (AII-8) and (AII-9) are used to take derivatives of covariant and contravariant vectors and can be generalized to take derivatives of covariant, contravariant, and mixed tensors.

Furthermore, since a basis vector $\vec{e}_i$ has a contravariant component e^i equal to one, with the use of the contravariant differential formula, equation (AII-4), one can show that

$$d\vec{e}_i = \Gamma^m_{il}\,\vec{e}_m dS^l \qquad \text{(AII-10)}$$

The above equation relating the change of the basis vector to a change of the curvilinear coordinate system is sometimes chosen as the defining formula for the Christoffel symbol instead of equation (AII-1).

One can also define Christoffel symbols of the first kind by the relation

$$\Gamma_{i,kl} = \gamma_{im}\Gamma^m_{kl} \qquad \text{(AII-11)}$$

from which one can easily obtain the inverse formula

$$\Gamma^i_{kl} = \gamma^{im}\,\Gamma_{m,kl} \quad . \qquad \text{(AII-12)}$$

It can be shown that the Christoffel symbols Γ^i_{kl} are symmetric in the indices k and l and hence the Christoffel symbols $\Gamma_{i,kl}$ are also symmetric in the indices k and l.

To calculate the Christoffel symbols one can follow the work of A. Lichnerowicz[22] and start with the definition of the metric tensor

$$\vec{e}_i \cdot \vec{e}_j = \gamma_{ij}. \qquad \text{(AII-13)}$$

By taking the differential of the above equation one obtains

$$d\vec{e}_i \cdot \vec{e}_j + \vec{e}_i \cdot d\vec{e}_j = d\gamma_{ij} \qquad \text{(AII-14)}$$

With the help of equations (AII-10), (AII-11), (AII-12) one gets

$$\Gamma_{i,kl} + \Gamma_{k,il} = \frac{\partial \gamma_{ik}}{\partial s^l} . \qquad \text{(AII-15)}$$

By cyclic permutation of the indices in the above equation one obtains the following two equations:

$$\Gamma_{i,kl} + \Gamma_{l,ik} = \frac{\partial \gamma_{il}}{\partial s^k} \qquad \text{(AII-15')}$$

and

$$-\Gamma_{l,ki} - \Gamma_{k,li} = -\frac{\partial \gamma_{kl}}{\partial s^i} . \qquad \text{(AII-15'')}$$

By adding the formulas (AII-15), (AII-15') and (AII-15'') and using the fact that $\Gamma_{i,kl}$ are symmetric in the indices k and l one obtains

$$\Gamma_{i,kl} = \frac{1}{2}\left(\frac{\partial \gamma_{ik}}{\partial s^l} + \frac{\partial \gamma_{il}}{\partial s^k} - \frac{\partial \gamma_{kl}}{\partial s^i}\right) . \qquad \text{(AII-16)}$$

From the above equation and equation (AII-12) one gets

$$\Gamma^i_{kl} = \frac{1}{2}\gamma^{im}\left(\frac{\partial \gamma_{mk}}{\partial s^l} + \frac{\partial \gamma_{il}}{\partial s^k} - \frac{\partial \gamma_{kl}}{\partial s^i}\right) . \qquad \text{(AII-16')}$$

Equations (AII-16) and (AII-16') are very useful since they permit the calculations of the Christoffel symbols from a knowledge of the body metric tensor.

With a knowledge of the Christoffel symbols one can examine the necessary and sufficient condition for a space to be Euclidian.

To do this one must pursue the concept of parallel displacement.

As an example consider two two-dimensional surfaces in fig. (10) One is Euclidian and is a plane, the other non-Euclidian, the surface of a sphere. The geodesics in the first are straight lines in the second arcs of great circles. If one considers a triangle in both surfaces and the parallel displacement of a vector R_1, the parallel displaced vector when one comes back to the starting point A is R_4. In the case of a Euclidian space $R_4 = R_1$, but this is not true for the non-Euclidian space. Therefore, it seems natural that the necessary and sufficient condition for a space to be Euclidian is the condition that the change in a vector after parallel displacement around a closed curve is zero.

Following again the arguments of Landau and Lifshitz [21] one can calculate the change in a parallel displacement of a vector over a closed curve by

$$\Delta A_k = \oint \delta A_k \quad . \tag{AII-17}$$

With the use of the parallel displacement part of equation (AII-6) or

$$\delta A_i = \Gamma^{\kappa}_{il} A_{\kappa} dS^{l} \tag{AII-18}$$

one obtains

$$\Delta A_k = \oint \Gamma^{i}_{\kappa l} A_i dS^{l} \tag{AII-19}$$

With the use of Stokes theorem, where df^{lm} is an antisymmetric tensor whose components are equal to the projected area of a surface element on the coordinate planes one obtains

$$\Delta A_k = \frac{1}{2}\int \left(\frac{\partial}{\partial s^l}(\Gamma^i_{km} A_i) - \frac{\partial}{\partial s^m}(\Gamma^i_{kl} A_i) \right) df^{lm}. \qquad \text{(AII-20)}$$

After some algebraic manipulations and making use of (AII-18) so that $\frac{\partial A_i}{\partial s^l} = \Gamma^n_{il} A_n$ one obtains

$$\Delta A_k = 1/2 \int R^i_{klm} A_i \, df^{lm} \qquad \text{(AII-21)}$$

where R^i_{klm} is called the curvature tensor or the Riemann-Christoffel tensor and is given by

$$R^i_{klm} = \frac{\partial \Gamma^i_{km}}{\partial s^l} - \frac{\partial \Gamma^i_{kl}}{\partial s^m} + \Gamma^i_{nl}\Gamma^n_{km} - \Gamma^i_{nm}\Gamma^n_{kl} \qquad \text{(AII-22)}$$

The fact that R^i_{klm} is a fourth order tensor is clear since ΔA_k are components of a covariant vectors, A_i the components of a covariant vector and df^{lm} the components of a second order contravariant tensor.

From (AII-21) we also see that when ΔA_k is zero since the surface elements are arbitrary, the curvature tensor must be zero,

$$R^i_{klm} = 0. \qquad \text{(AII-23)}$$

The above equation with the defining equation (AII-22) is the necessary and sufficient condition for a space to be Euclidian. The fact that it is necessary can easily be seen since for a Euclidian space the Christoffel symbols are zero. Hence, R^i_{klm} is also zero. Since R^i_{klm} are the components of a tensor they will be zero in any curvilinear coordinate system. The fact that the condition is sufficient is more complex and deals with the integrability of a system of equations (L.I.Sedov.[1]).

APPENDIX III

Divergence of the Stress Tensor

The condition for equilibrium of a small element of a material in static equilibrium in the absence of outside forces is given by

$$\oint \vec{f}\, dS = 0 \tag{AIII-1}$$

where f is the traction (force per unit area) on an element of surface dS. One would like to transform this integral to a volume integral. To do this one can take the scalar product of the above expression with an arbitrary constant covariant vector A (Sokolnikoff [(23)]). Hence,

$$\oint A_i f^i dS = 0. \tag{AIII-2}$$

Then from the expression for the traction in terms of the stress tensor and the covariant normal vector to the surface ν given by equation (II-31)

$$f^i = \pi^{ij} \nu_j \quad , \tag{AIII-3}$$

one obtains

$$\oint A_i \pi^{ij} \nu_j dS = 0. \tag{AIII-4}$$

This equation is of the form

$$\oint \vec{B}.\vec{dS} = \oint B^j \nu_j dS. \qquad \text{(AIII-5)}$$

From the divergence theorem it can be transformed to a volume integral

$$\oint B^j \nu_j dS = \int B^j{}_{,j} dV \qquad \text{(AIII-6)}$$

where in the above equation $B^j{}_{,j}$ is the divergence of the vector $\vec{B}$ obtained from the covariant derivative of the vector $\vec{B}$, $B^j{}_{,k}$ by summing over the index k by taking $k = j$.

Therefore, from equations (AIII-4) and (AIII-6) one obtains

$$\int (A_i \pi^{ij})_{,j} \, dV = 0. \qquad \text{(AIII-7)}$$

Since the covariant derivative of a product is obtained by the same rules as the derivatives of a product, one gets

$$(A_i \pi^{ij})_{,j} = A_{i,j} \pi^{ij} + A_i \pi^{ij}{}_{,j} \, . \qquad \text{(AIII-8)}$$

Since A is a constant vector $A_{i,j} = 0$ and one obtains by substituting equation (AIII-8) into equation (AIII-7)

$$A_i \int \pi^{ij}{}_{,j} \, dV = 0. \qquad \text{(AIII-9)}$$

The vector $\vec{A}$ is arbitrary, and the volume of integration is also arbitrary so that

$$\pi^{ij}{}_{,j} = 0. \qquad \text{(AIII-10)}$$

To find the expression for the divergence of the contravariant tensor π^{ij} one can take a simple case of a tensor formed by the direct product of two vectors

$$\pi^{ij} = A^i B^j. \qquad \text{(AIII-11)}$$

Taking the covariant derivative of the above expression one obtains

$$\pi^{ij}{}_{,k} = A^i{}_{,k} B^j + A^i B^j{}_{,k}. \qquad \text{(AIII-12)}$$

From the expression for the covariant derivatives of a vector given by equation (AII-8) of Appendix II one obtains

$$\pi^{ij}{}_{,k} = \left(\frac{\partial A^i}{\partial S^k} + \Gamma^i_{nk} A^n\right) B^j + A^i \left(\frac{\partial B^j}{\partial S^k} + \Gamma^j_{mk} B^m\right) \qquad \text{(AIII-13)}$$

or

$$\pi^{ij}{}_{,k} = \frac{\partial A^i B^j}{\partial S^k} + \Gamma^i_{nk} A^n B^j + \Gamma^j_{mk} A^i B^m. \qquad \text{(AIII-14)}$$

generalizing the above equation to any tensor one gets;

$$\pi^{ij}{}_{,k} = \frac{\partial \pi^{ij}}{\partial S^k} + \Gamma^i_{nk} \pi^{nj} + \Gamma^j_{mk} \pi^{im}. \qquad \text{(AIII-15)}$$

Hence the expression for the divergence becomes

$$\pi^{ij}{}_{,j} = \frac{\partial \Pi^{ij}}{\partial s^j} + \Gamma^i_{nj}\, \Pi^{nj} + \Gamma^j_{mj}\, \Pi^{im} \, . \qquad \text{(AIII-16)}$$

The above expression for the divergence of the stress tensor is useful for calculations of the divergence once the Christoffel symbols are known.

APPENDIX IV

Programs E5 and E5' and graphs of the radius, torque, and force

The programs E5 and E5' are very similar. The first one calculates the new radius, the dimensionless torque, and the dimensionless force at the end of the cylinder and plots them as a function of the dimensionless torsion angle. The second calculates and plots the torque and the force at the end of the cylinder in units of 10^{11} dynes-cm and 10^{11} dynes respectively.

The graphs for each material include the new radius, the dimensionless torque, the dimensionless force, the torque, and the force,as a function of the dimensionless torsion angle. The values of these quantities are printed on the side of each graph.

```
      PROGRAM E5
      COMMON  C1,Q1,Q2,Q3,Q4
   DIMENSION CCX(100),TM(100),BB(100),F(100)
   DIMENSION CX(100),B(100),PHI(100)
      READ*,Q1,Q2,Q3,Q4
      READ*,N
      READ*,PHIMIN,PHIMAX,M
      DO 20 J=1,M
      PHI(J)=PHIMIN+(PHIMAX+PHIMIN)*(REAL(J-1))/REAL(M-1)
C PRINT*,PHI(J)
      C1=(-1.0/8.0)*(Q1-2.0+4.0*Q3-Q2)/(2.0+Q1)
      CX(1)=1.0
      DO 10 I=1,N
      B(I)=FB(CX(I))
      CX(I+1)=FCX(B(I),PHI(J),CX(I))
C PRINT*,CX(I+1),B(I)
     IF(ABS(CX(I+1)-CX(I)).LT.0.000001)THEN
C PRINT*,CX(I+1)
   CCX(J)=CX(I+1)
   BB(J)=B(I)
   TM(J)=FTM(BB(J),PHI(J),CCX(J))
 F(J)=FF (BB(J),PHI(J),CCX(J))
     GO TO 20
     END IF
      10 CONTINUE
      20 CONTINUE
   CALL PLOT1(PHI,CCX,M)
   CALL PLOT1(PHI,TM,M)
 CALL PLOT1(PHI,F,M)
      STOP
      END
      FUNCTION FB(CX)
      COMMON  C1,Q1,Q2,Q3,Q4
      FB=(((-2.0*Q3)-(6.0*C1)-(0.5)*Q1*(8.0*C1+1.0))*(CX**2))/
+ (2.0+2.0*Q1)
      RETURN
      END
      FUNCTION  FCX(B,PHI,CX)
      COMMON  C1,Q1,Q2,Q3,Q4
      FCX=1.0+(PHI**2 )*(B*CX+C1*CX**3)
      RETURN
   END
   FUNCTION FTM(BB,PHI,CCX)
   COMMON C1,Q1,Q2,Q3,Q4
 FTMMU=(CCX**4)*(1.0+2.0*PHI**2*BB+(4.0*C1/3.0+(2.0/3.0))*(PHI**2)
+ *(CCX**2))
FTMLA =2.0*Q1*PHI**2*CCX**4*(BB+(8.0*C1+1.0)*CCX**2/6.0)
FTMA=4*Q2*PHI**2*CCX**4*((0.5*BB+(2.0*C1+2)*PHI**2/6.0)+
+ 4*PHI**4*Q2*(BB**2+2.0*BB*(2.0*C1+1)*CCX**2/3.0+(2.0*C1+1)**2*
+ CCX**4/8.0))
FTMB =4.0*Q3*PHI**2*CCX**4*(CCX**2/3.0+PHI**2*(2.0*BB**2+
+ (40.0*C1**2+4.0*C1+1.0)
+ *CCX**4/8.0+2.0*BB*(8.0*C1+1.0)*CCX**2/3.0))
FTM2B=8.0*Q3*PHI**4*CCX**4*(2.0*BB**2+BB*(4.0*C1+1.0)*CCX**2+
+(8.0*C1+1.0)*(2.0*C1+1.0)*CCX**4/8.0)
+ +8*Q3*PHI**2*CCX**4*(BB+(8.0*C1+1.0)*CCX**2/6.0)
FTMC=4.0*Q4*PHI**4*CCX**4*(4.0*BB**2+BB*4.0*(8.0*C1+1.0 )*PHI**2/3.0+
+ (8.0*C1+1)**2*PHI**4/8.0)
FTM=FTMMU+FTMLA+FTMA+FTMB+FTM2B+FTMC
RETURN
END
 FUNCTION FF(BB,PHI,CCX)
 COMMON C1,Q1,Q2,Q3,Q4
 FF01=PHI*CCX**4+Q1*PHI*CCX**2*(4.0*BB+(8.0*C1+1.0)*CCX**2/2.0)
FF2=Q2*PHI*CCX**4*(1.0+PHI**2*(2.0*BB+(2.0*C1+1.0)*PHI**2*2.0/3.0))
 FF31=0.5*Q3*PHI*CCX**2*(4.0*CCX**2+PHI**2*(32.0*BB**2+8.0*
+ (8.0*C1+1.0)*BB*CCX**2+4.0*(40.0*C1**2+4.0*C1 +1.0)*CCX**4/3.0))
 FF32=2.0*Q3*PHI**3*CCX**4*(4.0*BB+(8.0*C1+1.0)*CCX**2*2.0/3.0)
 FF4=Q4*PHI**3*(32.0*BB**2*CCX**2+8.0*BB*(8.0*C1+1.0)*CCX**4+
+ 2.0*(8.0*C1+1.0)**2*CCX**6/3.0)
FF=FF01+FF2+FF31+FF32+FF4
 RETURN
 END
```

```
        PROGRAM E5'
        COMMON  C1,Q1,Q2,Q3,Q4
 DIMENSION TMO(100),FO(100)
     DIMENSION CCX(100),TM(100),BB(100),F(100)
     DIMENSION CX(100),B(100),PHI(100)
 REAL K,MU
        READ*,Q1,Q2,Q3,Q4
        READ*,N
        READ*,PHIMIN,PHIMAX,M
        READ*,MU
        K=1.5708*MU
        DO 20 J=1,M
        PHI(J)=PHIMIN+(PHIMAX+PHIMIN)*(REAL(J-1))/REAL(M-1)
C PRINT*,PHI(J)
        C1=(-1.0/8.0)*(Q1-2.0+4.0*Q3-Q2)/(2.0+Q1)
        CX(1)=1.0
        DO 10 I=1,N
        B(I)=FB(CX(I))
        CX(I+1)=FCX(B(I),PHI(J),CX(I))
C PRINT*,CX(I+1),B(I)
       IF(ABS(CX(I+1)-CX(I)).LT.0.00001)THEN
C PRINT*,CX(I+1)
     CCX(J)=CX(I+1)
     BB(J)=B(I)
    TMO(J)=FTM(BB(J),PHI(J),CCX(J))
 TM(J)=TMO(J)*PHI(J)*K
 FO(J)=FF (BB(J),PHI(J),CCX(J))
 F(J)=FO(J)*PHI(J)*K
       GO TO 20
       END IF
        10 CONTINUE
        20 CONTINUE
     CALL PLOT1(PHI,CCX,M)
     CALL PLOT1(PHI,TM,M)
  CALL PLOT1(PHI,F,M)
        STOP
        END
        FUNCTION FB(CX)
        COMMON  C1,Q1,Q2,Q3,Q4
        FB=(((-2.0*Q3)-(6.0*C1)-(0.5)*Q1*(8.0*C1+1.0))*(CX**2))/
+ (2.0+2.0*Q1)
        RETURN
        END
        FUNCTION  FCX(B,PHI,CX)
        COMMON  C1,Q1,Q2,Q3,Q4
        FCX=1.0+(PHI**2 )*(B*CX+C1*CX**3)
        RETURN
     END
     FUNCTION FTM(BB,PHI,CCX)
     COMMON C1,Q1,Q2,Q3,Q4
   FTMMU=(CCX**4)*(1.0+2.0*PHI**2*BB+(4.0*C1/3.0+(2.0/3.0))*(PHI**2)
+ *(CCX**2))
 FTMLA =2.0*Q1*PHI**2*CCX**4*(BB+(8.0*C1+1.0)*CCX**2/6.0)
 FTMA=4*Q2*PHI**2*CCX**4*((0.5*BB+(2.0*C1+2)*PHI**2/6.0)+
+ 4*PHI**4*Q2*(BB**2+2.0*BB*(2.0*C1+1)*CCX**2/3.0+(2.0*C1+1)**2*
+ CCX**4/8.0))
 FTMB =4.0*Q3*PHI**2*CCX**4*(CCX**2/3.0+PHI**2*(2.0*BB**2+
+ (40.0*C1**2+4.0*C1+1.0)
+ *CCX**4/8.0+2.0*BB*(8.0*C1+1.0)*CCX**2/3.0))
 FTM2B=8.0*Q3*PHI**4*CCX**4*(2.0*BB**2+BB*(4.0*C1+1.0)*CCX**2+
+(8.0*C1+1.0)*(2.0*C1+1.0)*CCX**4/8.0)
+ +8*Q3*PHI**2*CCX**4*(BB+(8.0*C1+1.0)*CCX**2/6.0)
 FTMC=4.0*Q4*PHI**4*CCX**4*(4.0*BB**2+BB*4.0*(8.0*C1+1.0 )*PHI**2/3.0+
+ (8.0*C1+1)**2*PHI**4/8.0)
 FTM=FTMMU+FTMLA+FTMA+FTMB+FTM2B+FTMC
 RETURN
 END
  FUNCTION FF(BB,PHI,CCX)
  COMMON C1,Q1,Q2,Q3,Q4
  FF01=PHI*CCX**4+Q1*PHI*CCX**2*(4.0*BB+(8.0*C1+1.0)*CCX**2/2.0)
 FF2=Q2*PHI*CCX**4*(1.0+PHI**2*(2.0*BB+(2.0*C1+1.0)*PHI**2*2.0/3.0))
  FF31=0.5*Q3*PHI*CCX**2*(4.0*CCX**2+PHI**2*(32.0*BB**2+8.0*
+ (8.0*C1+1.0)*BB*CCX**2+4.0*(40.0*C1**2+4.0*C1 +1.0)*CCX**4/3.0))
  FF32=2.0*Q3*PHI**3*CCX**4*(4.0*BB+(8.0*C1+1.0)*CCX**2*2.0/3.0)
  FF4=Q4*PHI**3*(32.0*BB**2*CCX**2+8.0*BB*(8.0*C1+1.0)*CCX**4+
+  2.0*(8.0*C1+1.0)**2*CCX**6/3.0)
 FF=FF01+FF2+FF31+FF32+FF4
  RETURN
  END
```

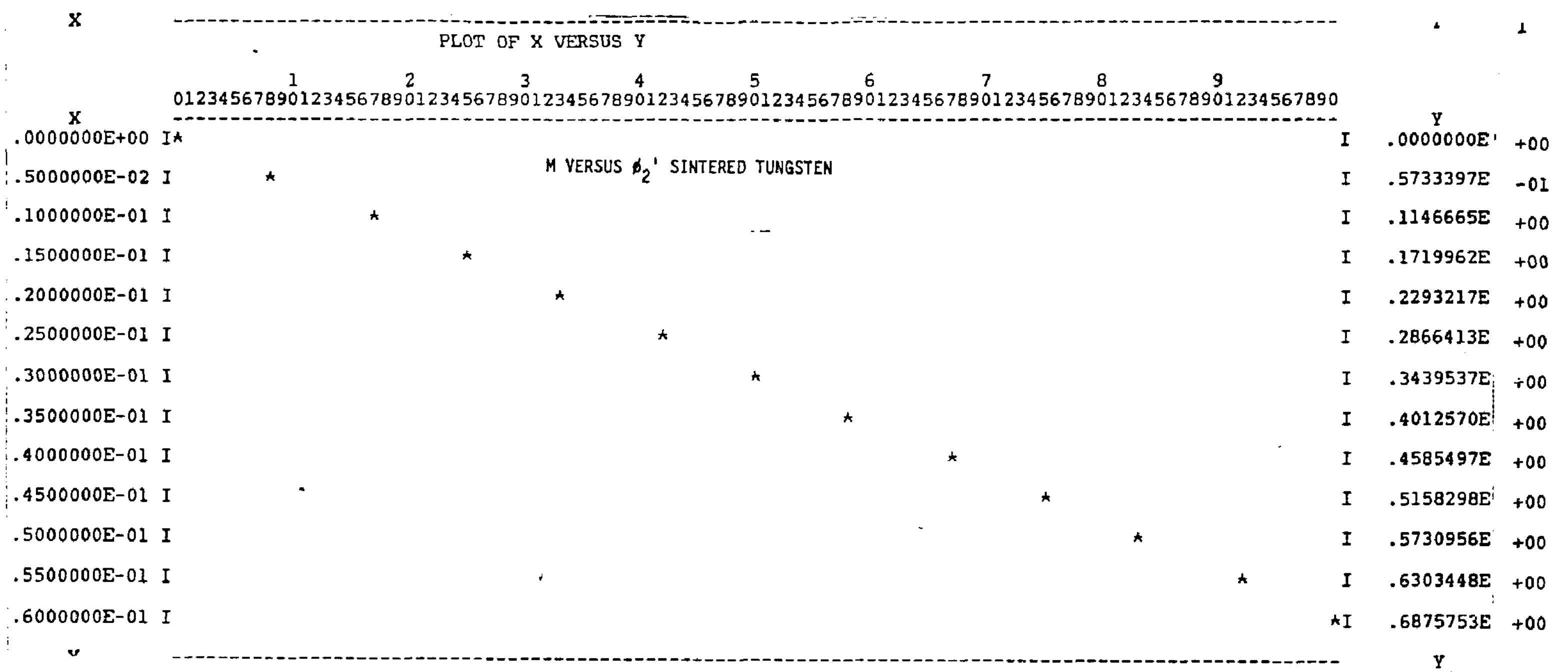
PLOT OF X VERSUS Y

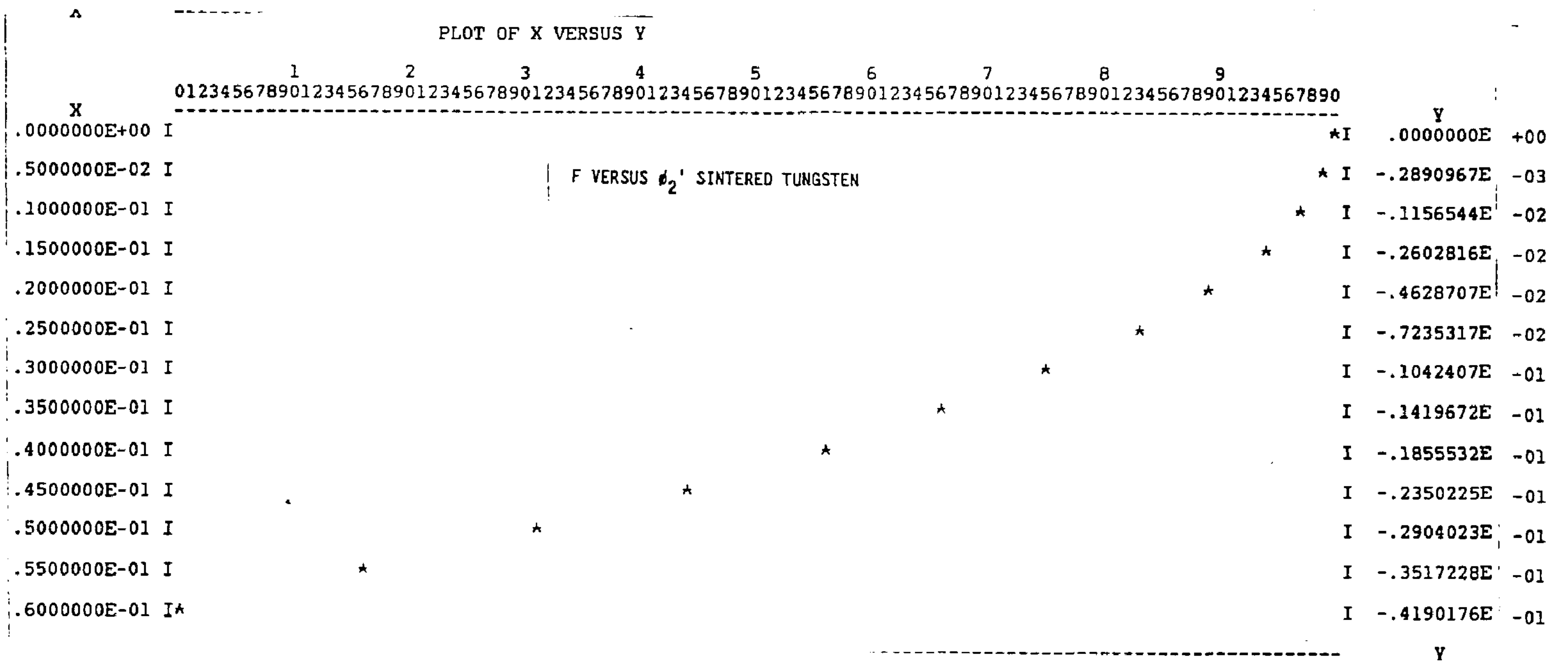
$S^3{}_o'$ VERSUS ϕ_2' SINTERED TUNGSTEN

```
                        1         2         3         4         5         6         7         8         9
              01234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
     X        -----------------------------------------------------------------------------------------------------      Y
.0000000E+00 I*                                                                                                    I  .1000000E +01
.5000000E-02 I *                                                                                                   I  .1000001E +01
.1000000E-01 I   *                                                                                                 I  .1000004E +01
.1500000E-01 I      *                                                                                              I  .1000009E +01
.2000000E-01 I           *                                                                                         I  .1000016E +01
.2500000E-01 I                 *                                                                                   I  .1000025E +01
.3000000E-01 I                         *                                                                           I  .1000036E +01
.3500000E-01 I                                  *                                                                  I  .1000049E +01
.4000000E-01 I                                            *                                                        I  .1000063E +01
.4500000E-01 I                                                        *                                            I  .1000080E +01
.5000000E-01 I                                                                     *                               I  .1000099E +01
.5500000E-01 I                                                                                    *                I  .1000120E +01
.6000000E-01 I                                                                                                    *I  .1000143E +01
```

```
                        PLOT OF X VERSUS Y

                        1         2         3         4         5         6         7         8         9
              01234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
      X       -----------------------------------------------------------------------------------------------------        Y
.0000000E+00 I                                                                                                    *I  .1000000E +01
                                             M' VERSUS ϕ2' SINTERED TUNGSTEN
.5000000E-02 I                                                                                                   * I  .9999959E +00
.1000000E-01 I                                                                                                 *   I  .9999836E +00
.1500000E-01 I                                                                                               *     I  .9999631E +00
.2000000E-01 I                                                                                           *         I  .9999341E +00
.2500000E-01 I                                                                                     *               I  .9998966E +00
.3000000E-01 I                                                                             *                       I  .9998502E +00
.3500000E-01 I                                                                     *                               I  .9997947E +00
.4000000E-01 I                                                         *                                           I  .9997298E +00
.4500000E-01 I                                             *                                                       I  .9996551E +00
.5000000E-01 I                                *                                                                    I  .9995702E +00
.5500000E-01 I                 *                                                                                   I  .9994745E +00
.6000000E-01 I*                                                                                                    I  .9993676E +00
              -----------------------------------------------------------------------------------------------------        Y
```

```
                                       PLOT OF X VERSUS Y

                        1         2         3         4         5         6         7         8         9
              0123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
      X                                                                                                                                         Y
              -------------------------------------------------------------------------------------------------------------------------
.0000000E+00 I                                                                                                                        *I   .0000000E +00
                                       F' VERSUS ϕ2' SINTERED TUNGSTEN
.5000000E-02 I                                                                                                              *          I  -.5042308E -02
.1000000E-01 I                                                                                                   *                     I  -.1008599E -01
.1500000E-01 I                                                                                          *                              I  -.1513243E -01
.2000000E-01 I                                                                                *                                        I  -.2018301E -01
.2500000E-01 I                                                                       *                                                 I  -.2523910E -01
.3000000E-01 I                                                            *                                                            I  -.3030208E -01
.3500000E-01 I                                                  *                                                                      I  -.3537336E -01
.4000000E-01 I                                         *                                                                               I  -.4045430E -01
.4500000E-01 I                              *                                                                                          I  -.4554632E -01
.5000000E-01 I                     *                                                                                                   I  -.5065080E -01
.5500000E-01 I          *                                                                                                              I  -.5576916E -01
.6000000E-01 I*                                                                                                                        I  -.6090280E -01
```


PLOT OF X VERSUS Y
M VERSUS ϕ_2' SINTERED TUNGSTEN
X
Y
.0000000E+00
.5000000E-02
.1000000E-01
.1500000E-01
.2000000E-01
.2500000E-01
.3000000E-01
.3500000E-01
.4000000E-01
.4500000E-01
.5000000E-01
.5500000E-01
.6000000E-01
.0000000E +00
.5733397E -01
.1146665E +00
.1719962E +00
.2293217E +00
.2866413E +00
.3439537E +00
.4012570E +00
.4585497E +00
.5158298E +00
.5730956E +00
.6303448E +00
.6875753E +00


PLOT OF X VERSUS Y
1 2 3 4 5 6 7 8 9
01234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
X
Y
F VERSUS ϕ_2' SINTERED TUNGSTEN
.0000000E+00 I *I .0000000E +00
.5000000E-02 I * I -.2890967E -03
.1000000E-01 I * I -.1156544E -02
.1500000E-01 I * I -.2602816E -02
.2000000E-01 I * I -.4628707E -02
.2500000E-01 I * I -.7235317E -02
.3000000E-01 I * I -.1042407E -01
.3500000E-01 I * I -.1419672E -01
.4000000E-01 I * I -.1855532E -01
.4500000E-01 I * I -.2350225E -01
.5000000E-01 I * I -.2904023E -01
.5500000E-01 I * I -.3517228E -01
.6000000E-01 I* I -.4190176E -01
Y

$S^3{}_o$' VERSUS ϕ_2' RESINTERED TUNGSTEN

```
                                   PLOT OF X VERSUS Y

                        1         2         3         4         5         6         7         8         9
              0123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
     X        -------------------------------------------------------------------------------------------        Y
.0000000E+00 I*                                                                                          I .1000000E +01
.5000000E-02 I *                                                                                         I .1000001E +01
.1000000E-01 I   *                                                                                       I .1000004E +01
.1500000E-01 I      *                                                                                    I .1000008E +01
.2000000E-01 I          *                                                                                I .1000015E +01
.2500000E-01 I                *                                                                          I .1000023E +01
.3000000E-01 I                       *                                                                   I .1000034E +01
.3500000E-01 I                               *                                                           I .1000046E +01
.4000000E-01 I                                        *                                                  I .1000060E +01
.4500000E-01 I                                                   *                                       I .1000075E +01
.5000000E-01 I                                                               *                           I .1000093E +01
.5500000E-01 I                                                                            *              I .1000113E +01
.6000000E-01 I                                                                                          *I .1000134E +01
     X        -------------------------------------------------------------------------------------------        Y
```

```
                                         PLOT OF X VERSUS Y

                        1         2         3         4         5         6         7         8         9
              01234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
     X                                                                                                                      Y
              -----------------------------------------------------------------------------------------------------
.0000000E+00 I                                                                                                    *I  .1000000E +01
.5000000E-02 I                             M' VERSUS ø2' RESINTERED TUNGSTEN                                     * I  .9999979E +00
.1000000E-01 I                                                                                                  *  I  .9999914E +00
.1500000E-01 I                                                                                               *     I  .9999805E +00
.2000000E-01 I                                                                                          *          I  .9999651E +00
.2500000E-01 I                                                                                     *               I  .9999450E +00
.3000000E-01 I                                                                             *                       I  .9999199E +00
.3500000E-01 I                                                                     *                               I  .9998896E +00
.4000000E-01 I                                                           *                                         I  .9998536E +00
.4500000E-01 I                                               *                                                     I  .9998117E +00
.5000000E-01 I                                 *                                                                   I  .9997632E +00
.5500000E-01 I                  *                                                                                  I  .9997078E +00
.6000000E-01 I*                                                                                                    I  .9996448E +00
              -----------------------------------------------------------------------------------------------------
```

```
  X           -----------------------------------------------------------------------------------------------------
                                   PLOT OF X VERSUS Y

                        1         2         3         4         5         6         7         8         9
              01234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
  X           -----------------------------------------------------------------------------------------------------      Y
.0000000E+00 I                                                                                                    *I   .0000000E +00
.5000000E-02 I                                F' VERSUS ø2' RESINTERED TUNGSTEN                          *         I  -.5620638E -02
.1000000E-01 I                                                                                   *                 I  -.1124262E -01
.1500000E-01 I                                                                           *                         I  -.1686731E -01
.2000000E-01 I                                                                   *                                 I  -.2249604E -01
.2500000E-01 I                                                           *                                         I  -.2813018E -01
.3000000E-01 I                                                  *                                                  I  -.3377108E -01
.3500000E-01 I                                          *                                                          I  -.3942009E -01
.4000000E-01 I                                  *                                                                  I  -.4507858E -01
.4500000E-01 I                         *                                                                           I  -.5074792E -01
.5000000E-01 I                 *                                                                                   I  -.5642948E -01
.5500000E-01 I        *                                                                                            I  -.6212464E -01
.6000000E-01 I*                                                                                                    I  -.6783478E -01
  X           -----------------------------------------------------------------------------------------------------      Y
```

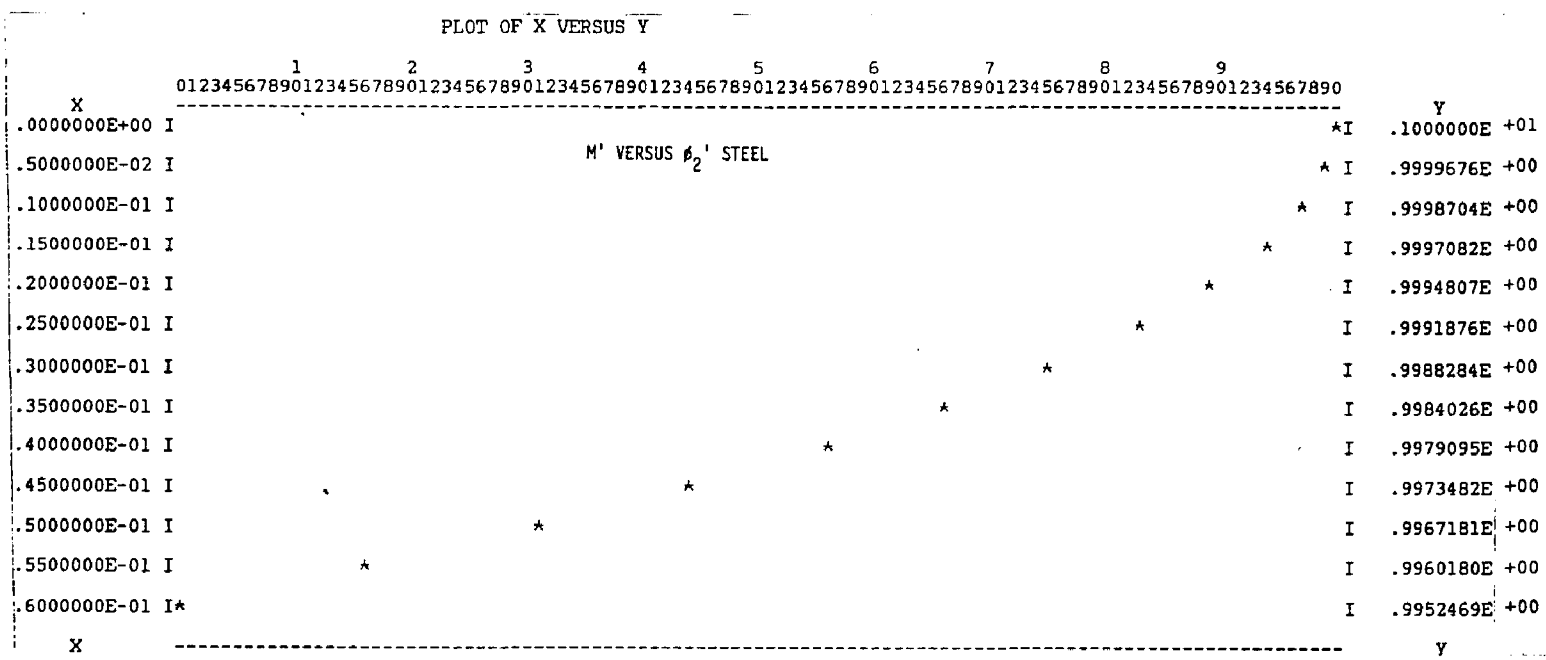
```
                 -----------------------------------------------------------------------------------------------------
                                       PLOT OF X VERSUS Y

                           1         2         3         4         5         6         7         8         9
                 01234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
      X          -----------------------------------------------------------------------------------------------------        Y
.0000000E+00 I*                                                                                                    I   .0000000E  +00
.5000000E-02 I        *                     M VERSUS ϕ2' RESINTERED TUNGSTEN                                       I   .1075996E  +00
.1000000E-01 I                 *                                                                                   I   .2151977E  +00
.1500000E-01 I                         *                                                                           I   .3227931E  +00
.2000000E-01 I                                 *                                                                   I   .4303842E  +00
.2500000E-01 I                                          *                                                          I   .5379694E  +00
.3000000E-01 I                                                  *                                                  I   .6455471E  +00
.3500000E-01 I                                                          *                                          I   .7531154E  +00
.4000000E-01 I                                                                   *                                 I   .8606724E  +00
.4500000E-01 I                                                                            *                        I   .9682158E  +00
.5000000E-01 I                                                                                    *                I   .1075743E  +01
.5500000E-01 I                                                                                             *       I   .1183252E  +01
.6000000E-01 I                                                                                                    *I   .1290739E  +01
      X          -----------------------------------------------------------------------------------------------------        Y
```

PLOT OF X VERSUS Y

F VERSUS ϕ_2' RESINTERED TUNGSTEN

```
                        1         2         3         4         5         6         7         8         9
              01234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
     X        -----------------------------------------------------------------------------------------------------      Y
.0000000E+00 I                                                                                                    *I  .0000000E +00
.5000000E-02 I                                                                                                   * I -.6047795E -03
.1000000E-01 I                                                                                                 *   I -.2419409E -02
.1500000E-01 I                                                                                              *      I -.5444760E -02
.2000000E-01 I                                                                                         *           I -.9682279E -02
.2500000E-01 I                                                                                   *                 I -.1513401E -01
.3000000E-01 I                                                                            *                        I -.2180257E -01
.3500000E-01 I                                                                   *                                 I -.2969116E -01
.4000000E-01 I                                                         *                                           I -.3880357E -01
.4500000E-01 I                                             *                                                       I -.4914419E -01
.5000000E-01 I                                *                                                                    I -.6071801E -01
.5500000E-01 I                *                                                                                    I -.7353058E -01
.6000000E-01 I*                                                                                                    I -.8758810E -01
     v        -----------------------------------------------------------------------------------------------------      v
```

```
                                       PLOT OF X VERSUS Y

                        1         2         3         4         5         6         7         8         9
              01234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
     X        -----------------------------------------------------------------------------------------------------          Y
.0000000E+00 I*                                                                                                    I  .1000000E +01
                                           S3o' VERSUS φ2' STEEL
.5000000E-02 I *                                                                                                   I  .1000003E +01
.1000000E-01 I   *                                                                                                 I  .1000012E +01
.1500000E-01 I      *                                                                                              I  .1000027E +01
.2000000E-01 I           *                                                                                         I  .1000047E +01
.2500000E-01 I                 *                                                                                   I  .1000074E +01
.3000000E-01 I                         *                                                                           I  .1000107E +01
.3500000E-01 I                                  *                                                                  I  .1000145E +01
.4000000E-01 I                                            *                                                        I  .1000190E +01
.4500000E-01 I                                                        *                                            I  .1000240E +01
.5000000E-01 I                                                                     *                               I  .1000297E +01
.5500000E-01 I                                                                                    *                I  .1000359E +01
.6000000E-01 I                                                                                                    *I  .1000427E +01
     X        -----------------------------------------------------------------------------------------------------          Y
```


PLOT OF X VERSUS Y
M' VERSUS ø2' STEEL
X
Y
.0000000E+00
.5000000E-02
.1000000E-01
.1500000E-01
.2000000E-01
.2500000E-01
.3000000E-01
.3500000E-01
.4000000E-01
.4500000E-01
.5000000E-01
.5500000E-01
.6000000E-01
.1000000E +01
.9999676E +00
.9998704E +00
.9997082E +00
.9994807E +00
.9991876E +00
.9988284E +00
.9984026E +00
.9979095E +00
.9973482E +00
.9967181E +00
.9960180E +00
.9952469E +00

```
                                     PLOT OF X VERSUS Y

                        1         2         3         4         5         6         7         8         9
              01234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
     X        -----------------------------------------------------------------------------------------------------        Y
.0000000E+00 I                                                                                                    *I   .0000000E  +00
.5000000E-02 I                                     F' VERSUS ø2' STEEL                                    *        I  -.7759458E  -02
.1000000E-01 I                                                                                    *                I  -.1552300E  -01
.1500000E-01 I                                                                           *                         I  -.2329472E  -01
.2000000E-01 I                                                                   *                                 I  -.3107871E  -01
.2500000E-01 I                                                           *                                         I  -.3887907E  -01
.3000000E-01 I                                                  *                                                  I  -.4669990E  -01
.3500000E-01 I                                          *                                                          I  -.5454533E  -01
.4000000E-01 I                                  *                                                                  I  -.6241948E  -01
.4500000E-01 I                         *                                                                           I  -.7032649E  -01
.5000000E-01 I                 *                                                                                   I  -.7827053E  -01
.5500000E-01 I         *                                                                                           I  -.8625578E  -01
.6000000E-01 I*                                                                                                    I  -.9428643E  -01
```

```
                                PLOT OF X VERSUS Y

                        1         2         3         4         5         6         7         8         9
              01234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
     X        -----------------------------------------------------------------------------------------------------     Y
.0000000E+00 I*                                                                                                    I  .0000000E +00
.5000000E-02 I        *                          M VERSUS ϕ2' STEEL                                                I  .6447925E -01
.1000000E-01 I                 *                                                                                   I  .1289460E +00
.1500000E-01 I                         *                                                                           I  .1933876E +00
.2000000E-01 I                                 *                                                                   I  .2577914E +00
.2500000E-01 I                                          *                                                          I  .3221448E +00
.3000000E-01 I                                                  *                                                  I  .3864348E +00
.3500000E-01 I                                                           *                                         I  .4506484E +00
.4000000E-01 I                                                                   *                                 I  .5147723E +00
.4500000E-01 I                                                                           *                         I  .5787932E +00
.5000000E-01 I                                                                                   *                 I  .6426972E +00
.5500000E-01 I                                                                                            *        I  .7064703E +00
.6000000E-01 I                                                                                                    *I  .7700983E +00
     X        -----------------------------------------------------------------------------------------------------     Y
```

PLOT OF X VERSUS Y

F VERSUS ϕ_2' STEEL

```
                        1         2         3         4         5         6         7         8         9
              01234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
     X        -----------------------------------------------------------------------------------------------------       Y
.0000000E+00 I                                                                                                    *I   .0000000E  +00
.5000000E-02 I                                                                                                   * I  -.5003407E  -03
.1000000E-01 I                                                                                                 *   I  -.2001888E  -02
.1500000E-01 I                                                                                              *      I  -.4506225E  -02
.2000000E-01 I                                                                                         *           I  -.8015988E  -02
.2500000E-01 I                                                                                   *                 I  -.1253487E  -01
.3000000E-01 I                                                                           *                         I  -.1806763E  -01
.3500000E-01 I                                                                  *                                  I  -.2462009E  -01
.4000000E-01 I                                                        *                                            I  -.3219913E  -01
.4500000E-01 I                                            *                                                        I  -.4081272E  -01
.5000000E-01 I                               *                                                                     I  -.5046989E  -01
.5500000E-01 I                *                                                                                    I  -.6118077E  -01
.6000000E-01 I*                                                                                                    I  -.7295658E  -01
```

```
                                  PLOT OF X VERSUS Y

                        1         2         3         4         5         6         7         8         9
              01234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
      X       -----------------------------------------------------------------------------------------------------       Y
.0000000E+00 I*                                                                                                    I   .1000000E +01
                                  S3o' VERSUS ø2' SINTERED MOLYBDENUM
.5000000E-02 I *                                                                                                   I   .1000001E +01
.1000000E-01 I   *                                                                                                 I   .1000005E +01
.1500000E-01 I      *                                                                                              I   .1000010E +01
.2000000E-01 I           *                                                                                         I   .1000018E +01
.2500000E-01 I                 *                                                                                   I   .1000029E +01
.3000000E-01 I                         *                                                                           I   .1000041E +01
.3500000E-01 I                                  *                                                                  I   .1000056E +01
.4000000E-01 I                                            *                                                        I   .1000074E +01
.4500000E-01 I                                                        *                                            I   .1000093E +01
.5000000E-01 I                                                                     *                               I   .1000115E +01
.5500000E-01 I                                                                                    *                I   .1000139E +01
.6000000E-01 I                                                                                                    *I   .1000165E +01
      X       -----------------------------------------------------------------------------------------------------       Y
```

```
                              PLOT OF X VERSUS Y

                        1         2         3         4         5         6         7         8         9
              01234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
     X        -----------------------------------------------------------------------------------------------------       Y
.0000000E+00 I                                                                                                    *I  .1000000E  +01
.5000000E-02 I                               M' VERSUS φ2' SINTERED MOLYBDENUM                                   * I  .9999887E  +00
.1000000E-01 I                                                                                                 *   I  .9999546E  +00
.1500000E-01 I                                                                                              *      I  .9998977E  +00
.2000000E-01 I                                                                                         *           I  .9998179E  +00
.2500000E-01 I                                                                                   *                 I  .9997148E  +00
.3000000E-01 I                                                                            *                        I  .9995884E  +00
.3500000E-01 I                                                                   *                                 I  .9994382E  +00
.4000000E-01 I                                                        *                                            I  .9992638E  +00
.4500000E-01 I                                            *                                                        I  .9990649E  +00
.5000000E-01 I                               *                                                                     I  .9988409E  +00
.5500000E-01 I                *                                                                                    I  .9985912E  +00
.6000000E-01 I*                                                                                                    I  .9983153E  +00
```

```
    ^         -----------------------------------------------------------------------------------------------------          Y       I
                                     PLOT OF X VERSUS Y

                        1         2         3         4         5         6         7         8         9
              01234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
    X         -----------------------------------------------------------------------------------------------------          Y
.0000000E+00 I                                                                                                    *I   .0000000E  +00
                                              F' VERSUS Ø2' SINTERED MOLYBDENUM
.5000000E-02 I                                                                                            *        I  -.5306471E  -02
.1000000E-01 I                                                                                   *                 I  -.1061442E  -01
.1500000E-01 I                                                                           *                         I  -.1592533E  -01
.2000000E-01 I                                                                   *                                 I  -.2124067E  -01
.2500000E-01 I                                                           *                                         I  -.2656194E  -01
.3000000E-01 I                                                  *                                                  I  -.3189062E  -01
.3500000E-01 I                                          *                                                          I  -.3722819E  -01
.4000000E-01 I                                  *                                                                  I  -.4257615E  -01
.4500000E-01 I                         *                                                                           I  -.4793600E  -01
.5000000E-01 I                 *                                                                                   I  -.5330924E  -01
.5500000E-01 I        *                                                                                            I  -.5869738E  -01
.6000000E-01 I*                                                                                                    I  -.6410194E  -01
    X         -----------------------------------------------------------------------------------------------------          Y
```

```
                                PLOT OF X VERSUS Y

                        1         2         3         4         5         6         7         8         9
              0123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890123456789
     X        -----------------------------------------------------------------------------------------------------        Y
.0000000E+00 I*                                                                                                     I   .0000000E  +00
.5000000E-02 I        *                                                                                             I   .8639302E  -01
                                          M VERSUS ϕ2' SINTERED MOLYBDENUM
.1000000E-01 I                  *                                                                                   I   .1727802E  +00
.1500000E-01 I                          *                                                                           I   .2591555E  +00
.2000000E-01 I                                  *                                                                   I   .3455131E  +00
.2500000E-01 I                                           *                                                          I   .4318468E  +00
.3000000E-01 I                                                   *                                                  I   .5181506E  +00
.3500000E-01 I                                                           *                                          I   .6044182E  +00
.4000000E-01 I                                                                    *                                 I   .6906432E  +00
.4500000E-01 I                                                                            *                         I   .7768189E  +00
.5000000E-01 I                                                                                    *                 I   .8629386E  +00
.5500000E-01 I                                                                                             *        I   .9489952E  +00
.6000000E-01 I                                                                                                     *I   .1034981E  +01
```

F VERSUS ϕ_2' SINTERED MOLYBDENUM

```
      X         -----------------------------------------------------------------------------------------------------
                                       PLOT OF X VERSUS Y
                          1         2         3         4         5         6         7         8         9
                0123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890123456789 0
      X         -----------------------------------------------------------------------------------------------------        Y
.0000000E+00 I                                                                                                     *I   .0000000E +00
.5000000E-02 I                                                                                                   * I  -.4584472E -03
.1000000E-01 I                                                                                                 *   I  -.1834044E -02
.1500000E-01 I                                                                                               *     I  -.4127558E -02
.2000000E-01 I                                                                                          *          I  -.7340267E -02
.2500000E-01 I                                                                                    *                I  -.1147396E -01
.3000000E-01 I                                                                            *                        I  -.1653095E -01
.3500000E-01 I                                                                   *                                 I  -.2251405E -01
.4000000E-01 I                                                         *                                           I  -.2942659E -01
.4500000E-01 I                                             *                                                       I  -.3727245E -01
.5000000E-01 I                               *                                                                     I  -.4605599E -01
.5500000E-01 I                *                                                                                    I  -.5578212E -01
.6000000E-01 I*                                                                                                    I  -.6645627E -01
```

PLOT OF X VERSUS Y

```
                        1         2         3         4         5         6         7         8         9
     X        0123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890      Y
              ------------------------------------------------------------------------------------------------------------------------
.0000000E+00 I*                                                                                                                       I  .1000000E +01
.5000000E-02 I *                               S^3o' VERSUS ø_2' RESINTERED MOLYBDENUM                                                I  .1000002E +01
.1000000E-01 I    *                                                                                                                   I  .1000008E +01
.1500000E-01 I       *                                                                                                                I  .1000019E +01
.2000000E-01 I             *                                                                                                          I  .1000033E +01
.2500000E-01 I                    *                                                                                                   I  .1000051E +01
.3000000E-01 I                              *                                                                                         I  .1000074E +01
.3500000E-01 I                                        *                                                                               I  .1000101E +01
.4000000E-01 I                                                    *                                                                   I  .1000132E +01
.4500000E-01 I                                                                   *                                                    I  .1000167E +01
.5000000E-01 I                                                                                  *                                     I  .1000206E +01
.5500000E-01 I                                                                                                    *                   I  .1000249E +01
.6000000E-01 I                                                                                                                       *I  .1000297E +01
```

```
     X       -----------------------------------------------------------------------------------------------------         Y    1
                                  PLOT OF X VERSUS Y

                       1         2         3         4         5         6         7         8         9
              01234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
     X        -----------------------------------------------------------------------------------------------------         Y
.0000000E+00 I                                                                                                    *I   .1000000E +01
.5000000E-02 I                                  M' VERSUS Ø2' RESINTERED MOLYBDENUM                              * I   .9999760E +00
.1000000E-01 I                                                                                                 *   I   .9999039E +00
.1500000E-01 I                                                                                              *      I   .9997836E +00
.2000000E-01 I                                                                                         *           I   .9996149E +00
.2500000E-01 I                                                                                   *                 I   .9993976E +00
.3000000E-01 I                                                                           *                         I   .9991312E +00
.3500000E-01 I                                                                  *                                  I   .9988154E +00
.4000000E-01 I                                                        *                                            I   .9984496E +00
.4500000E-01 I                                            *                                                        I   .9980332E +00
.5000000E-01 I                               *                                                                     I   .9975657E +00
.5500000E-01 I                *                                                                                    I   .9970461E +00
.6000000E-01 I*                                                                                                    I   .9964738E +00
```

-127-

```
        X     -----------------------------------------------------------------------------------------------------         Y      1
                                   PLOT OF X VERSUS Y

                        1         2         3         4         5         6         7         8         9
              01234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
        X     -----------------------------------------------------------------------------------------------------         Y
.0000000E+00 I                                                                                                   *I   .0000000E  +00
.5000000E-02 I                             F' VERSUS ϕ2' RESINTERED MOLYBDENUM                           *        I  -.6221428E  -02
.1000000E-01 I                                                                                  *                 I  -.1244488E  -01
.1500000E-01 I                                                                          *                         I  -.1867238E  -01
.2000000E-01 I                                                                  *                                 I  -.2490596E  -01
.2500000E-01 I                                                          *                                         I  -.3114765E  -01
.3000000E-01 I                                                 *                                                  I  -.3739949E  -01
.3500000E-01 I                                         *                                                          I  -.4366350E  -01
.4000000E-01 I                                 *                                                                  I  -.4994175E  -01
.4500000E-01 I                        *                                                                           I  -.5623627E  -01
.5000000E-01 I                *                                                                                   I  -.6254914E  -01
.5500000E-01 I       *                                                                                            I  -.6888242E  -01
.6000000E-01 I*                                                                                                   I  -.7523819E  -01
```

```
                                PLOT OF X VERSUS Y

                        1         2         3         4         5         6         7         8         9
              01234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
      X       -----------------------------------------------------------------------------------------------------          Y
.0000000E+00 I*                                                                                                    I   .0000000E +00
.5000000E-02 I        *                         M VERSUS Ø2' RESINTERED MOLYBDENUM                                 I   .9738726E -01
.1000000E-01 I                 *                                                                                   I   .1947605E +00
.1500000E-01 I                         *                                                                           I   .2921057E +00
.2000000E-01 I                                 *                                                                   I   .3894086E +00
.2500000E-01 I                                          *                                                          I   .4866551E +00
.3000000E-01 I                                                  *                                                  I   .5838307E +00
.3500000E-01 I                                                          *                                          I   .6809209E +00
.4000000E-01 I                                                                   *                                 I   .7779108E +00
.4500000E-01 I                                                                           *                         I   .8747853E +00
.5000000E-01 I                                                                                   *                 I   .9715290E +00
.5500000E-01 I                                                                                            *        I   .1068126E +01
.6000000E-01 I                                                                                                    *I   .1164561E +01
      V       -----------------------------------------------------------------------------------------------------          V
```

```
                                  PLOT OF X VERSUS Y

                        1         2         3         4         5         6         7         8         9
     X        01234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890         Y
.0000000E+00 I                                                                                                   *I   .0000000E +00
.5000000E-02 I                               F VERSUS ϕ2' RESINTERED MOLYBDENUM                                 * I  -.6024657E -03
.1000000E-01 I                                                                                                 *  I  -.2410257E -02
.1500000E-01 I                                                                                              *     I  -.5424545E -02
.2000000E-01 I                                                                                         *          I  -.9647302E -02
.2500000E-01 I                                                                                   *                I  -.1508128E -01
.3000000E-01 I                                                                           *                        I  -.2173001E -01
.3500000E-01 I                                                                  *                                 I  -.2959783E -01
.4000000E-01 I                                                        *                                           I  -.3868988E -01
.4500000E-01 I                                            *                                                       I  -.4901208E -01
.5000000E-01 I                               *                                                                    I  -.6057116E -01
.5500000E-01 I                *                                                                                   I  -.7337467E -01
.6000000E-01 I*                                                                                                   I  -.8743096E -01
```

PLOT OF X VERSUS Y

$S^{3}{}_{o}'$ VERSUS ϕ_2' MAGNESIUM

```
                        1         2         3         4         5         6         7         8         9
             01234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
    X        -----------------------------------------------------------------------------------------------------          Y
.0000000E+00 I*                                                                                                   I   .1000000E +01
.5000000E-02 I *                                                                                                  I   .1000003E +01
.1000000E-01 I   *                                                                                                I   .1000012E +01
.1500000E-01 I      *                                                                                             I   .1000027E +01
.2000000E-01 I           *                                                                                        I   .1000047E +01
.2500000E-01 I                 *                                                                                  I   .1000074E +01
.3000000E-01 I                         *                                                                          I   .1000106E +01
.3500000E-01 I                                  *                                                                 I   .1000145E +01
.4000000E-01 I                                            *                                                       I   .1000189E +01
.4500000E-01 I                                                        *                                           I   .1000239E +01
.5000000E-01 I                                                                      *                             I   .1000296E +01
.5500000E-01 I                                                                                     *              I   .1000358E +01
.6000000E-01 I                                                                                                   *I   .1000426E +01
```

-130-

```
     X       -----------------------------------------------------------------------------------------------------        Y       1
                                PLOT OF X VERSUS Y

                       1         2         3         4         5         6         7         8         9
             01234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
     X       -----------------------------------------------------------------------------------------------------        Y
.0000000E+00 I                                                                                                    *I   .1000000E  +01
.5000000E-02 I                                       M' VERSUS ø2' MAGNESIUM                                     * I   .9999674E  +00
.1000000E-01 I                                                                                                 *   I   .9998694E  +00
.1500000E-01 I                                                                                              *      I   .9997060E  +00
.2000000E-01 I                                                                                          *          I   .9994768E  +00
.2500000E-01 I                                                                                    *                I   .9991815E  +00
.3000000E-01 I                                                                            *                        I   .9988195E  +00
.3500000E-01 I                                                                   *                                 I   .9983903E  +00
.4000000E-01 I                                                         *                                           I   .9978932E  +00
.4500000E-01 I                                            *                                                        I   .9973273E  +00
.5000000E-01 I                               *                                                                     I   .9966917E  +00
.5500000E-01 I                *                                                                                    I   .9959855E  +00
.6000000E-01 I*                                                                                                    I   .9952074E  +00
```

F' VERSUS ϕ_2' MAGNESIUM

```
     X       -----------------------------------------------------------------------------------------------------        Y     1
                                   PLOT OF X VERSUS Y
                        1         2         3         4         5         6         7         8         9
              01234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
     X        -----------------------------------------------------------------------------------------------------        Y
.0000000E+00 I                                                                                                    *I    .0000000E +00
.5000000E-02 I                                                                                            *        I   -.8663967E -02
.1000000E-01 I                                                                                    *                I   -.1733205E -01
.1500000E-01 I                                                                           *                         I   -.2600837E -01
.2000000E-01 I                                                                   *                                 I   -.3469705E -01
.2500000E-01 I                                                           *                                         I   -.4340222E -01
.3000000E-01 I                                                  *                                                  I   -.5212802E -01
.3500000E-01 I                                          *                                                          I   -.6087859E -01
.4000000E-01 I                                  *                                                                  I   -.6965810E -01
.4500000E-01 I                         *                                                                           I   -.7847072E -01
.5000000E-01 I                 *                                                                                   I   -.8732065E -01
.5500000E-01 I        *                                                                                            I   -.9621210E -01
.6000000E-01 I*                                                                                                    I   -.1051493E +00
     X        -----------------------------------------------------------------------------------------------------        Y
```

```
     X        ------------------------------------------------------------
                                    PLOT OF X VERSUS Y

                        1         2         3         4         5         6         7         8         9
              01234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
     X        -----------------------------------------------------------------------------------------------------       Y
.0000000E+00 I*                                                                                                    I  .0000000E +00
.5000000E-02 I        *                            M VERSUS φ2' MAGNESIUM                                          I  .1303721E -01
.1000000E-01 I                 *                                                                                   I  .2607188E -01
.1500000E-01 I                         *                                                                           I  .3910142E -01
.2000000E-01 I                                 *                                                                   I  .5212328E -01
.2500000E-01 I                                          *                                                          I  .6513484E -01
.3000000E-01 I                                                  *                                                  I  .7813350E -01
.3500000E-01 I                                                           *                                         I  .9111657E -01
.4000000E-01 I                                                                   *                                 I  .1040814E +00
.4500000E-01 I                                                                           *                         I  .1170251E +00
.5000000E-01 I                                                                                   *                 I  .1299451E +00
.5500000E-01 I                                                                                            *        I  .1428383E +00
.6000000E-01 I                                                                                                    *I  .1557019E +00
```

```
     X        -----------------------------------------------------------------------------------------------------
                                  PLOT OF X VERSUS Y

                        1         2         3         4         5         6         7         8         9
              01234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
     X        -----------------------------------------------------------------------------------------------------        Y
.0000000E+00 I                                                                                                    *I   .0000000E +00
.5000000E-02 I                                    F VERSUS ϕ2' MAGNESIUM                                        *  I  -.1129579E -03
.1000000E-01 I                                                                                                *    I  -.4519381E -03
.1500000E-01 I                                                                                              *      I  -.1017263E -02
.2000000E-01 I                                                                                         *           I  -.1809471E -02
.2500000E-01 I                                                                                   *                 I  -.2829313E -02
.3000000E-01 I                                                                           *                         I  -.4077758E -02
.3500000E-01 I                                                                  *                                  I  -.5555992E -02
.4000000E-01 I                                                        *                                            I  -.7265418E -02
.4500000E-01 I                                            *                                                        I  -.9207657E -02
.5000000E-01 I                               *                                                                     I  -.1138455E -01
.5500000E-01 I                *                                                                                    I  -.1379817E -01
.6000000E-01 I*                                                                                                    I  -.1645079E -01
```

```
                              PLOT OF X VERSUS Y

                        1         2         3         4         5         6         7         8         9
              0123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
     X        ---------------------------------------------------------------------------------------------------    Y
.0000000E+00 I*                                                                                                   I  .1000000E +01
                                     S3o' VERSUS ϕ2' ALUMINIUM
.5000000E-02 I *                                                                                                  I  .1000007E +01
.1000000E-01 I   *                                                                                                I  .1000028E +01
.1500000E-01 I      *                                                                                             I  .1000062E +01
.2000000E-01 I           *                                                                                        I  .1000110E +01
.2500000E-01 I                 *                                                                                  I  .1000173E +01
.3000000E-01 I                         *                                                                          I  .1000249E +01
.3500000E-01 I                                  *                                                                 I  .1000339E +01
.4000000E-01 I                                            *                                                       I  .1000442E +01
.4500000E-01 I                                                        *                                           I  .1000560E +01
.5000000E-01 I                                                                     *                              I  .1000692E +01
.5500000E-01 I                                                                                    *               I  .1000837E +01
.6000000E-01 I                                                                                                   *I  .1000997E +01
                   ---------------------------------------------------------------------------------    Y
```

```
      ^       ------------------------------------------------------------------------------------------------------------------------
                                         PLOT OF X VERSUS Y

                          1           2           3           4           5           6           7           8           9
              0123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
      X       ------------------------------------------------------------------------------------------------------------------------      Y
.0000000E+00 I                                                                                                                       *I  .1000000E+01
.5000000E-02 I                                              M' VERSUS ø2' ALUMINIUM                                                 * I  .9998443E+00
.1000000E-01 I                                                                                                                    *   I  .9993770E+00
.1500000E-01 I                                                                                                                *       I  .9985975E+00
.2000000E-01 I                                                                                                          *             I  .9975047E+00
.2500000E-01 I                                                                                                   *                    I  .9960970E+00
.3000000E-01 I                                                                                          *                             I  .9943727E+00
.3500000E-01 I                                                                               *                                        I  .9923293E+00
.4000000E-01 I                                                                   *                                                    I  .9899640E+00
.4500000E-01 I                                                     *                                                                  I  .9872737E+00
.5000000E-01 I                                     *                                                                                  I  .9842548E+00
.5500000E-01 I                   *                                                                                                    I  .9809032E+00
.6000000E-01 I*                                                                                                                       I  .9772146E+00
      v       ------------------------------------------------------------------------------------------------------------------------      v
```

```
                                     PLOT OF X VERSUS Y

                         1         2         3         4         5         6         7         8         9
              01234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
     X                                                                                                                         Y
.0000000E+00 I                                                                                                   *I   .0000000E+00
.5000000E-02 I                                                                                           *        I  -.1477569E-01
.1000000E-01 I                                      F' VERSUS Ø2' ALUMINIUM                      *                I  -.2956825E-01
.1500000E-01 I                                                                           *                        I  -.4439459E-01
.2000000E-01 I                                                                   *                                I  -.5927161E-01
.2500000E-01 I                                                           *                                        I  -.7421629E-01
.3000000E-01 I                                                   *                                                I  -.8924565E-01
.3500000E-01 I                                           *                                                        I  -.1043768E+00
.4000000E-01 I                                  *                                                                 I  -.1196269E+00
.4500000E-01 I                          *                                                                         I  -.1350133E+00
.5000000E-01 I                 *                                                                                  I  -.1505532E+00
.5500000E-01 I         *                                                                                          I  -.1662643E+00
.6000000E-01 I*                                                                                                   I  -.1821644E+00
```

PLOT OF X VERSUS Y

M VERSUS ϕ_2' ALUMINIUM

```
                        1         2         3         4         5         6         7         8         9
              0123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
      X       ----------------------------------------------------------------------------------------------------             Y
.0000000E+00 I*                                                                                                   I   .0000000E +00
.5000000E-02 I         *                                                                                          I   .2167366E -01
.1000000E-01 I                 *                                                                                  I   .4332701E -01
.1500000E-01 I                          *                                                                         I   .6493972E -01
.2000000E-01 I                                  *                                                                 I   .8649133E -01
.2500000E-01 I                                          *                                                         I   .1079613E +00
.3000000E-01 I                                                   *                                                I   .1293288E +00
.3500000E-01 I                                                           *                                        I   .1505729E +00
.4000000E-01 I                                                                    *                               I   .1716722E +00
.4500000E-01 I                                                                            *                       I   .1926053E +00
.5000000E-01 I                                                                                    *               I   .2133500E +00
.5500000E-01 I                                                                                            *       I   .2338842E +00
.6000000E-01 I                                                                                                   *I   .2541848E +00
```

```
     Λ        -----------------------------------------------------------------------------------------------------     Y   1
                                     PLOT OF X VERSUS Y

                        1         2         3         4         5         6         7         8         9
              01234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
     X        -----------------------------------------------------------------------------------------------------      Y
.0000000E+00 I                                                                                                   *I   .0000000E +00
.5000000E-02 I                                        F VERSUS φ2' ALUMINIUM                                    * I  -.3204371E -03
.1000000E-01 I                                                                                                *   I  -.1282473E -02
.1500000E-01 I                                                                                             *      I  -.2888314E -02
.2000000E-01 I                                                                                        *           I  -.5141629E -02
.2500000E-01 I                                                                                  *                 I  -.8047562E -02
.3000000E-01 I                                                                           *                        I  -.1161274E -01
.3500000E-01 I                                                                  *                                 I  -.1584528E -01
.4000000E-01 I                                                       *                                            I  -.2075481E -01
.4500000E-01 I                                            *                                                       I  -.2635244E -01
.5000000E-01 I                               *                                                                    I  -.3265085E -01
.5500000E-01 I                *                                                                                   I  -.3966424E -01
.6000000E-01 I*                                                                                                   I  -.4740839E -01
     V                                                                                                                   V
```

PLOT OF X VERSUS Y

S^3o' VERSUS ϕ_2' COLUMBIUM

```
                        1         2         3         4         5         6         7         8         9
              01234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
     X        -----------------------------------------------------------------------------------------------------          Y
.0000000E+00 I                                                                                                   *I  .1000000E +01
.5000000E-02 I                                                                                                  * I  .1000000E +01
.1000000E-01 I                                                                                                *   I  .9999999E +00
.1500000E-01 I                                                                                              *     I  .9999998E +00
.2000000E-01 I                                                                                         *          I  .9999997E +00
.2500000E-01 I                                                                                   *                I  .9999995E +00
.3000000E-01 I                                                                           *                        I  .9999993E +00
.3500000E-01 I                                                                  *                                 I  .9999991E +00
.4000000E-01 I                                                        *                                           I  .9999988E +00
.4500000E-01 I                                            *                                                       I  .9999984E +00
.5000000E-01 I                               *                                                                    I  .9999981E +00
.5500000E-01 I                *                                                                                   I  .9999977E +00
.6000000E-01 I*                                                                                                   I  .9999972E +00
                                                  ---------------------------------------------------                    Y
```

```
     X        ------------------------------------------------------
                                       PLOT OF X VERSUS Y

                        1         2         3         4         5         6         7         8         9
              0123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
     X        -----------------------------------------------------------------------------------------------------     Y
.0000000E+00 I                                                                                                    *I   .1000000E +01
.5000000E-02 I                                    M' VERSUS Ø2' COLUMBIUM                                       * I   .9998043E +00
.1000000E-01 I                                                                                               *   I   .9992171E +00
.1500000E-01 I                                                                                            *      I   .9982386E +00
.2000000E-01 I                                                                                       *           I   .9968685E +00
.2500000E-01 I                                                                                 *                 I   .9951070E +00
.3000000E-01 I                                                                         *                         I   .9929539E +00
.3500000E-01 I                                                                *                                  I   .9904093E +00
.4000000E-01 I                                                      *                                            I   .9874732E +00
.4500000E-01 I                                          *                                                        I   .9841454E +00
.5000000E-01 I                             *                                                                     I   .9804260E +00
.5500000E-01 I              *                                                                                    I   .9763149E +00
.6000000E-01 I*                                                                                                  I   .9718121E +00
     X        -----------------------------------------------------------------------------------------------------     Y
```

```
                              PLOT OF X VERSUS Y

                        1         2         3         4         5         6         7         8         9
              01234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
      X       -----------------------------------------------------------------------------------------------------        Y
.0000000E+00 I*                                                                                                   I  .0000000E +00
.5000000E-02 I         *                               F' VERSUS Ø2' COLUMBIUM                                    I  .1001381E -01
.1000000E-01 I                 *                                                                                  I  .2001805E -01
.1500000E-01 I                          *                                                                         I  .3000318E -01
.2000000E-01 I                                  *                                                                 I  .3995962E -01
.2500000E-01 I                                          *                                                         I  .4987784E -01
.3000000E-01 I                                                  *                                                 I  .5974826E -01
.3500000E-01 I                                                          *                                         I  .6956140E -01
.4000000E-01 I                                                                   *                                I  .7930807E -01
.4500000E-01 I                                                                           *                        I  .8897815E -01
.5000000E-01 I                                                                                    *               I  .985624E -01
.5500000E-01 I                                                                                           *        I  .1080513E +00
.6000000E-01 I                                                                                                   *I  .1174355E +00
```

```
     X          ------------------------------------------------------------------------------------------------------------------------       Y     1
                                           PLOT OF X VERSUS Y

                          1          2          3          4          5          6          7          8          9
                0123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
     X          ------------------------------------------------------------------------------------------------------------------------       Y
 .0000000E+00 I*                                                                                                                       I   .0000000E +00
 .5000000E-02 I          *                                         M VERSUS ∅2' COLUMBIUM                                              I   .2944674E -01
 .1000000E-01 I                   *                                                                                                    I   .5885889E -01
 .1500000E-01 I                              *                                                                                         I   .8820186E -01
 .2000000E-01 I                                       *                                                                                I   .1174411E +00
 .2500000E-01 I                                                  *                                                                     I   .1465419E +00
 .3000000E-01 I                                                           *                                                            I   .1754699E +00
 .3500000E-01 I                                                                     *                                                  I   .2041902E +00
 .4000000E-01 I                                                                                *                                       I   .2326684E +00
 .4500000E-01 I                                                                                         *                              I   .2608699E +00
 .5000000E-01 I                                                                                                   *                    I   .2887600E +00
 .5500000E-01 I                                                                                                            *           I   .3163040E +00
 .6000000E-01 I                                                                                                                       *I   .3434675E +00
     X          ------------------------------------------------------------------------------------------------------------------------       Y
```

```
                                     PLOT OF X VERSUS Y

                        1         2         3         4         5         6         7         8         9
              01234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
     X        -----------------------------------------------------------------------------------------------------         Y
.0000000E+00 I*                                                                                                    I  .0000000E +00
.5000000E-02 I *                                                                                                   I  .2949316E -03
.1000000E-01 I   *                                                                                                 I  .1179163E -02
.1500000E-01 I      *                                   F VERSUS ϕ2' COLUMBIUM                                     I  .2651006E -02
.2000000E-01 I           *                                                                                         I  .4707643E -02
.2500000E-01 I                  *                                                                                  I  .7345135E -02
.3000000E-01 I                         *                                                                           I  .1055842E -01
.3500000E-01 I                                   *                                                                 I  .1434130E -01
.4000000E-01 I                                             *                                                       I  .1868648E -01
.4500000E-01 I                                                         *                                           I  .2358551E -01
.5000000E-01 I                                                                      *                              I  .2902887E -01
.5500000E-01 I                                                                                    *                I  .3500588E -01
.6000000E-01 I                                                                                                    *I  .4150476E -01
```

PLOT OF X VERSUS Y

S^3o' VERSUS ϕ_2' PYREX

```
                        1         2         3         4         5         6         7         8         9
              01234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
     X        -----------------------------------------------------------------------------------------------------     Y
.0000000E+00 I                                                                                                    *I   .1000000E  +01
.5000000E-02 I                                                                                                   * I   .9999958E  +00
.1000000E-01 I                                                                                                 *   I   .9999830E  +00
.1500000E-01 I                                                                                              *      I   .9999619E  +00
.2000000E-01 I                                                                                         *           I   .9999322E  +00
.2500000E-01 I                                                                                   *                 I   .9998941E  +00
.3000000E-01 I                                                                           *                         I   .9998475E  +00
.3500000E-01 I                                                                  *                                  I   .9997924E  +00
.4000000E-01 I                                                        *                                            I   .9997289E  +00
.4500000E-01 I                                            *                                                        I   .9996570E  +00
.5000000E-01 I                               *                                                                     I   .9995766E  +00
.5500000E-01 I                *                                                                                    I   .9994878E  +00
.6000000E-01 I*                                                                                                    I   .9993907E  +00
```

```
     X         -----------------------------------------------------------------------------------------------------
                         PLOT OF X VERSUS Y

                         1         2         3         4         5         6         7         8         9
               01234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
     X         -----------------------------------------------------------------------------------------------------        Y
.0000000E+00 I                                                                                                   *I   .1000000E +01
.5000000E-02 I                                      M' VERSUS Ø2' PYREX                                         * I   .9997930E +00
.1000000E-01 I                                                                                                 *  I   .9991721E +00
.1500000E-01 I                                                                                              *     I   .9981379E +00
.2000000E-01 I                                                                                         *          I   .9966911E +00
.2500000E-01 I                                                                                   *                I   .9948329E +00
.3000000E-01 I                                                                           *                        I   .9925647E +00
.3500000E-01 I                                                                  *                                 I   .9898884E +00
.4000000E-01 I                                                       *                                            I   .9868059E +00
.4500000E-01 I                                            *                                                       I   .9833198E +00
.5000000E-01 I                              *                                                                     I   .9794328E +00
.5500000E-01 I                *                                                                                   I   .9751478E +00
.6000000E-01 I*                                                                                                   I   .9704683E +00
```

```
    X         -------------------------------------------------------------------------------------------------------------------------      Y      1
                                         PLOT OF X VERSUS Y

                          1           2           3           4           5           6           7           8           9
              0123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
    X         -------------------------------------------------------------------------------------------------------------------------      Y
.0000000E+00 I*                                                                                                                        I   .0000000E  +00
                                                           F' VERSUS Ø2' PYREX
.5000000E-02 I          *                                                                                                              I   .1294070E  -01
.1000000E-01 I                     *                                                                                                   I   .2587710E  -01
.1500000E-01 I                              *                                                                                          I   .3880488E  -01
.2000000E-01 I                                         *                                                                               I   .5171977E  -01
.2500000E-01 I                                                  *                                                                      I   .6461746E  -01
.3000000E-01 I                                                            *                                                            I   .7749369E  -01
.3500000E-01 I                                                                       *                                                 I   .9034423E  -01
.4000000E-01 I                                                                                *                                        I   .1031648E  +00
.4500000E-01 I                                                                                          *                              I   .1159513E  +00
.5000000E-01 I                                                                                                    *                    I   .1286995E  +00
.5500000E-01 I                                                                                                              *          I   .1414053E  +00
.6000000E-01 I                                                                                                                        *I   .1540644E  +00
```

```
                          PLOT OF X VERSUS Y

                          1         2         3         4         5         6         7         8         9
   X            0123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890123456789          Y
                ----------------------------------------------------------------------------------------------------
.0000000E+00 I*                                                                                                    I   .0000000E +00
                                                   M VERSUS ø2' PYREX
.5000000E-02 I         *                                                                                           I   .2159403E -01
.1000000E-01 I                 *                                                                                   I   .4316124E -01
.1500000E-01 I                          *                                                                          I   .6467484E -01
.2000000E-01 I                                  *                                                                  I   .8610813E -01
.2500000E-01 I                                           *                                                         I   .1074345E +00
.3000000E-01 I                                                   *                                                 I   .1286275E +00
.3500000E-01 I                                                            *                                        I   .1496607E +00
.4000000E-01 I                                                                    *                                I   .1705082E +00
.4500000E-01 I                                                                            *                        I   .1911441E +00
.5000000E-01 I                                                                                    *                I   .2115428E +00
.5500000E-01 I                                                                                            *        I   .2316790E +00
.6000000E-01 I                                                                                                    *I   .2515279E +00
   X            ----------------------------------------------------------------------------------------------------          Y
```

PLOT OF X VERSUS Y

```
                        1         2         3         4         5         6         7         8         9
              01234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
     X        -----------------------------------------------------------------------------------------------------      Y
.0000000E+00 I*                                                                                                    I  .0000000E  +00
.5000000E-02 I *                                      F VERSUS ϕ2' PYREX                                           I  .2794986E  -03
.1000000E-01 I   *                                                                                                 I  .1117813E  -02
.1500000E-01 I      *                                                                                              I  .2514382E  -02
.2000000E-01 I           *                                                                                         I  .4468277E  -02
.2500000E-01 I                 *                                                                                   I  .6978201E  -02
.3000000E-01 I                         *                                                                           I  .1004249E  -01
.3500000E-01 I                                  *                                                                  I  .1365910E  -01
.4000000E-01 I                                             *                                                       I  .1782565E  -01
.4500000E-01 I                                                        *                                            I  .2253937E  -01
.5000000E-01 I                                                                      *                              I  .2779717E  -01
.5500000E-01 I                                                                                    *                I  .3359557E  -01
.6000000E-01 I                                                                                                    *I  .3993076E  -01
     X        -----------------------------------------------------------------------------------------------------      Y
```

PLOT OF X VERSUS Y

$S^{3}{}_{0}$' VERSUS ϕ_{2}' FUSED SILICA

```
                        1         2         3         4         5         6         7         8         9
              01234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
      X       -----------------------------------------------------------------------------------------------------       Y
.0000000E+00 I                                                                                                    *I  .1000000E +01
.5000000E-02 I                                                                                                   * I  .9999921E +00
.1000000E-01 I                                                                                                 *   I  .9999684E +00
.1500000E-01 I                                                                                              *      I  .9999289E +00
.2000000E-01 I                                                                                         *           I  .9998737E +00
.2500000E-01 I                                                                                   *                 I  .9998027E +00
.3000000E-01 I                                                                           *                         I  .9997159E +00
.3500000E-01 I                                                                  *                                  I  .9996135E +00
.4000000E-01 I                                                       *                                             I  .9994953E +00
.4500000E-01 I                                            *                                                        I  .9993615E +00
.5000000E-01 I                              *                                                                      I  .9992121E +00
.5500000E-01 I                *                                                                                    I  .9990471E +00
.6000000E-01 I*                                                                                                    I  .9988666E +00
```

```
    X         -----------------------------------------------------------------------------------------------------          Y       1
                                     PLOT OF X VERSUS Y

                        1         2         3         4         5         6         7         8         9
              01234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
    X         -----------------------------------------------------------------------------------------------------          Y
.0000000E+00 I                                                                                                    *I   .1000000E +01
.5000000E-02 I                                      M' VERSUS φ2' FUSED SILICA                                   * I   .9999723E +00
.1000000E-01 I                                                                                                 *   I   .9998891E +00
.1500000E-01 I                                                                                              *      I   .9997507E +00
.2000000E-01 I                                                                                         *           I   .9995570E +00
.2500000E-01 I                                                                                  *                  I   .9993083E +00
.3000000E-01 I                                                                           *                         I   .9990047E +00
.3500000E-01 I                                                                  *                                  I   .9986467E +00
.4000000E-01 I                                                       *                                             I   .9982344E +00
.4500000E-01 I                                           *                                                         I   .9977684E +00
.5000000E-01 I                              *                                                                      I   .9972489E +00
.5500000E-01 I                *                                                                                    I   .9966765E +00
.6000000E-01 I*                                                                                                    I   .9960516E +00
```

F' VERSUS ϕ_2' FUSED SILICA

```
  X            -----------------------------------------------------------------------------------------------------      Y      1
                                  PLOT OF X VERSUS Y

                        1         2         3         4         5         6         7         8         9
              01234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
  X           -----------------------------------------------------------------------------------------------------       Y
.0000000E+00 I*                                                                                                    I  .0000000E  +00
.5000000E-02 I        *                                                                                            I  .5263003E  -02
.1000000E-01 I                 *                                                                                   I  .1052674E  -01
.1500000E-01 I                         *                                                                           I  .1579193E  -01
.2000000E-01 I                                 *                                                                   I  .2105932E  -01
.2500000E-01 I                                          *                                                          I  .2632961E  -01
.3000000E-01 I                                                  *                                                  I  .3160355E  -01
.3500000E-01 I                                                          *                                          I  .3688184E  -01
.4000000E-01 I                                                                   *                                 I  .4216520E  -01
.4500000E-01 I                                                                           *                         I  .4745427E  -01
.5000000E-01 I                                                                                   *                 I  .5274983E  -01
.5500000E-01 I                                                                                            *        I  .5805253E  -01
.6000000E-01 I                                                                                                    *I  .6336305E  -01
  --                                                         ------------------------------------------------------       Y
```

```
                                     PLOT OF X VERSUS Y

                        1         2         3         4         5         6         7         8         9
              01234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
     X        -----------------------------------------------------------------------------------------------------       Y
.0000000E+00 I*                                                                                                    I   .0000000E +00
.5000000E-02 I        *                             M VERSUS ø2' FUSED SILICA                                      I   .2450380E -01
.1000000E-01 I                 *                                                                                   I   .4900353E -01
.1500000E-01 I                         *                                                                           I   .7349511E -01
.2000000E-01 I                                 *                                                                   I   .9797450E -01
.2500000E-01 I                                          *                                                          I   .1224376E +00
.3000000E-01 I                                                  *                                                  I   .1468806E +00
.3500000E-01 I                                                          *                                          I   .1712992E +00
.4000000E-01 I                                                                   *                                 I   .1956897E +00
.4500000E-01 I                                                                           *                         I   .2200482E +00
.5000000E-01 I                                                                                   *                 I   .2443707E +00
.5500000E-01 I                                                                                            *        I   .2686534E +00
.6000000E-01 I                                                                                                    *I   .2928928E +00
     X        -----------------------------------------------------------------------------------------------------       Y
```

```
                                PLOT OF X VERSUS Y

                         1         2         3         4         5         6         7         8         9
               0123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
     X         ------------------------------------------------------------------------------------------------------------------------     Y
.0000000E+00 I*                                                                                                                       I  .0000000E +00
.5000000E-02 I  *                                            F VERSUS ϕ2' FUSED SILICA                                                I  .1289660E -03
.1000000E-01 I    *                                                                                                                   I  .5159044E -03
.1500000E-01 I        *                                                                                                               I  .1160919E -02
.2000000E-01 I             *                                                                                                          I  .2064190E -02
.2500000E-01 I                    *                                                                                                   I  .3225968E -02
.3000000E-01 I                              *                                                                                         I  .4646572E -02
.3500000E-01 I                                         *                                                                              I  .6326393E -02
.4000000E-01 I                                                    *                                                                   I  .8265891E -02
.4500000E-01 I                                                                   *                                                    I  .1046559E -01
.5000000E-01 I                                                                                  *                                     I  .1292610E -01
.5500000E-01 I                                                                                                    *                   I  .1564806E -01
.6000000E-01 I                                                                                                                       *I  .1863222E -01
                                           ------------------------------------------------------------------------------------         V
```

```
                              PLOT OF X VERSUS Y

                        1         2         3         4         5         6         7         8         9
              0123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
      X       -------------------------------------------------------------------------------------------------------------------------       Y
.0000000E+00 I*                                                                                                                        I .1000000E +01
.5000000E-02 I *                                      S3o' VERSUS Ø2' POLYSTERENE                                                      I .1000004E +01
.1000000E-01 I    *                                                                                                                    I .1000015E +01
.1500000E-01 I       *                                                                                                                 I .1000034E +01
.2000000E-01 I             *                                                                                                           I .1000060E +01
.2500000E-01 I                    *                                                                                                    I .1000094E +01
.3000000E-01 I                              *                                                                                          I .1000135E +01
.3500000E-01 I                                         *                                                                               I .1000184E +01
.4000000E-01 I                                                     *                                                                   I .1000241E +01
.4500000E-01 I                                                                   *                                                     I .1000305E +01
.5000000E-01 I                                                                                   *                                     I .1000377E +01
.5500000E-01 I                                                                                                     *                   I .1000456E +01
.6000000E-01 I                                                                                                                        *I .1000543E +01
      V                                 ---------------------------------------------------------------------------------------        V
```

```
                              PLOT OF X VERSUS Y

                        1         2         3         4         5         6         7         8         9
              01234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
     X        -----------------------------------------------------------------------------------------------------      Y
.0000000E+00 I                                                                                                   *I   .1000000E+01
.5000000E-02 I                                   M' VERSUS ϕ2' POLYSTERENE                                      * I   .9999147E+00
.1000000E-01 I                                                                                                *   I   .9996587E+00
.1500000E-01 I                                                                                             *      I   .9992317E+00
.2000000E-01 I                                                                                        *           I   .9986333E+00
.2500000E-01 I                                                                                  *                 I   .9978629E+00
.3000000E-01 I                                                                          *                         I   .9969196E+00
.3500000E-01 I                                                                 *                                  I   .9958025E+00
.4000000E-01 I                                                       *                                            I   .9945103E+00
.4500000E-01 I                                           *                                                        I   .9930418E+00
.5000000E-01 I                              *                                                                     I   .9913954E+00
.5500000E-01 I               *                                                                                    I   .9895694E+00
.6000000E-01 I*                                                                                                   I   .9875619E+00
```

F' VERSUS ϕ_2' POLYSTERENE

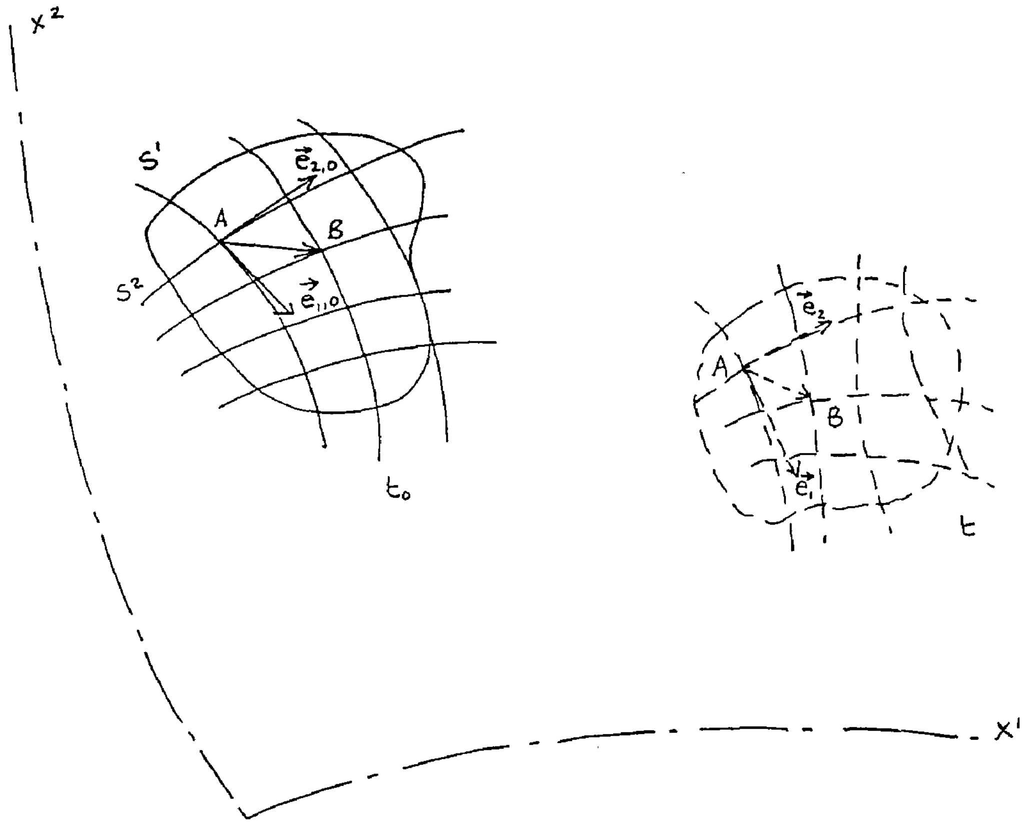
```
              -----------------------------------------------------------------------------------------------------
                                     PLOT OF X VERSUS Y

                        1         2         3         4         5         6         7         8         9
              01234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
     X        -----------------------------------------------------------------------------------------------------           Y
.0000000E+00 I                                                                                                    *I   .0000000E +00
.5000000E-02 I                                                                                             *       I  -.7536421E -02
.1000000E-01 I                                                                                     *               I  -.1508312E -01
.1500000E-01 I                                                                             *                       I  -.2265037E -01
.2000000E-01 I                                                                     *                               I  -.3024846E -01
.2500000E-01 I                                                            *                                        I  -.3788771E -01
.3000000E-01 I                                                    *                                                I  -.4557844E -01
.3500000E-01 I                                            *                                                        I  -.5333099E -01
.4000000E-01 I                                    *                                                                I  -.6115576E -01
.4500000E-01 I                           *                                                                         I  -.6906317E -01
.5000000E-01 I                  *                                                                                  I  -.7706366E -01
.5500000E-01 I         *                                                                                           I  -.8516776E -01
.6000000E-01 I*                                                                                                    I  -.9338601E -01
     X        -----------------------------------------------------------------------------------------------------           Y
```

```
                                   PLOT OF X VERSUS Y

                        1         2         3         4         5         6         7         8         9
              01234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
      X       -----------------------------------------------------------------------------------------------------      Y
.0000000E+00 I*                                                                                                    I   .0000000E +00
.5000000E-02 I        *                             M VERSUS Ø2' POLYSTERENE                                       I   .1083759E -02
.1000000E-01 I                 *                                                                                   I   .2166962E -02
.1500000E-01 I                         *                                                                           I   .3249051E -02
.2000000E-01 I                                  *                                                                  I   .4329465E -02
.2500000E-01 I                                          *                                                          I   .5407644E -02
.3000000E-01 I                                                  *                                                  I   .6483021E -02
.3500000E-01 I                                                           *                                         I   .7555023E -02
.4000000E-01 I                                                                   *                                 I   .8623075E -02
.4500000E-01 I                                                                           *                         I   .9686593E -02
.5000000E-01 I                                                                                    *                I   .1074498E -01
.5500000E-01 I                                                                                            *        I   .1179765E -01
.6000000E-01 I                                                                                                    *I   .1284397E -01
```

F VERSUS ϕ_2' POLYSTERENE

```
     X         -----------------------------------------------------------------------------------------------------        Y      1
                                     PLOT OF X VERSUS Y

                         1         2         3         4         5         6         7         8         9
               01234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
     X         -----------------------------------------------------------------------------------------------------        Y
.0000000E+00 I                                                                                                    *I   .0000000E +00
.5000000E-02 I                                                                                                   *  I  -.8175394E -05
.1000000E-01 I                                                                                                 *    I  -.3272387E -04
.1500000E-01 I                                                                                              *       I  -.7371246E -04
.2000000E-01 I                                                                                         *            I  -.1312529E -03
.2500000E-01 I                                                                                   *                  I  -.2055017E -03
.3000000E-01 I                                                                            *                         I  -.2966604E -03
.3500000E-01 I                                                                   *                                  I  -.4049755E -03
.4000000E-01 I                                                        *                                             I  -.5307389E -03
.4500000E-01 I                                             *                                                        I  -.6742878E -03
.5000000E-01 I                               *                                                                      I  -.8360054E -03
.5500000E-01 I                *                                                                                     I  -.1016321E -02
.6000000E-01 I*                                                                                                     I  -.1215709E -02
```

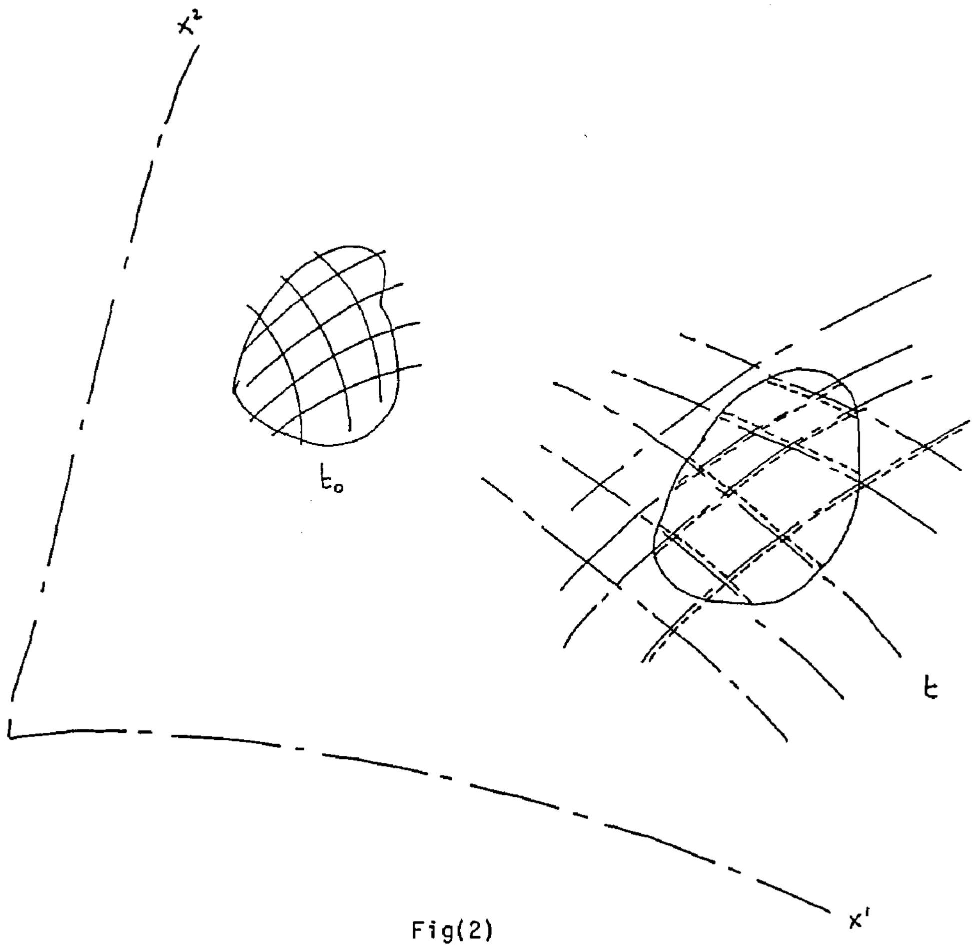
FIGURES



Fig(1)

Body Coordinate system

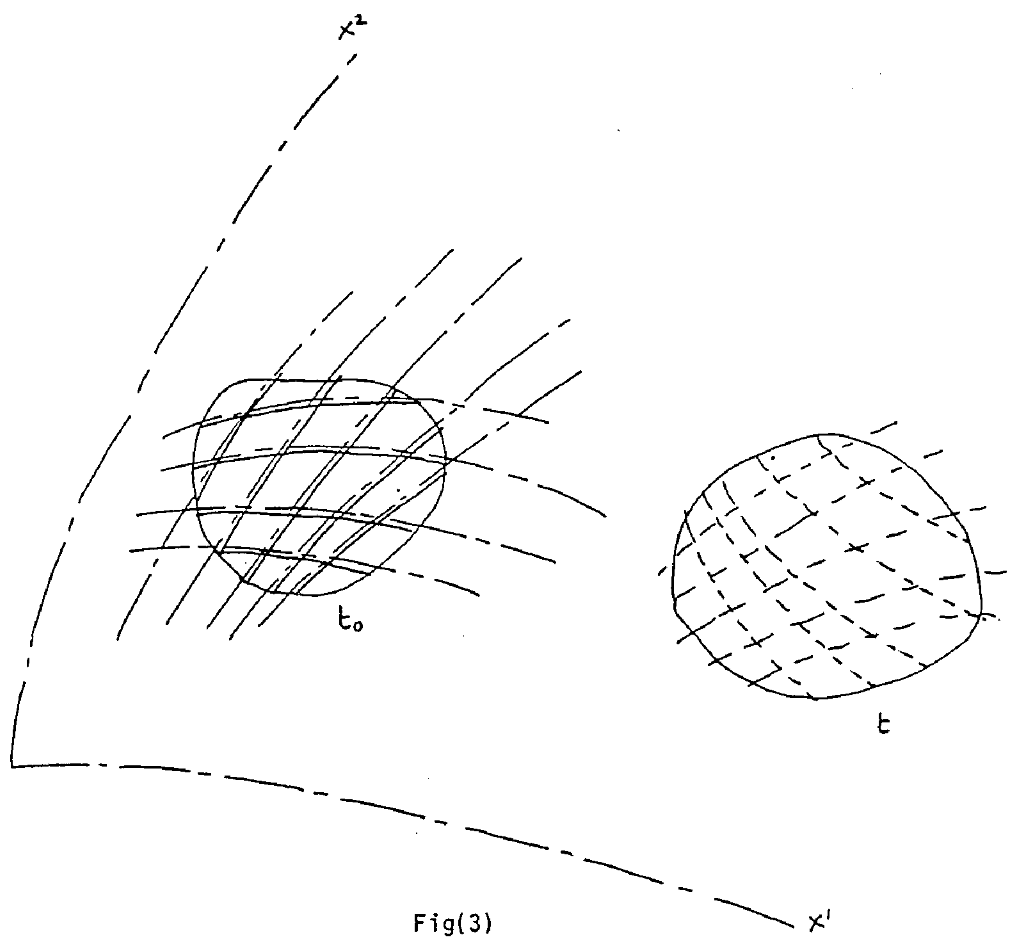
In this figure for the sake of simplicity one has assumed a two dimensional body. The dash-dot line represents the space coordinate system. The solid line represents the body coordinate system in state t_o and the dashed line the body coordinate system in state t.



Fig(2)

Isomorphism of the space and body coordinate systems in state t

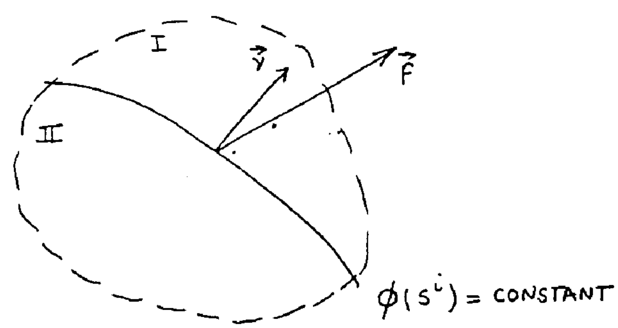
The dash-dot line represents the space coordinate system, the dashed line the body coordinate system at time t and the solid line the body coordinate system at time t_0.



Fig(3)

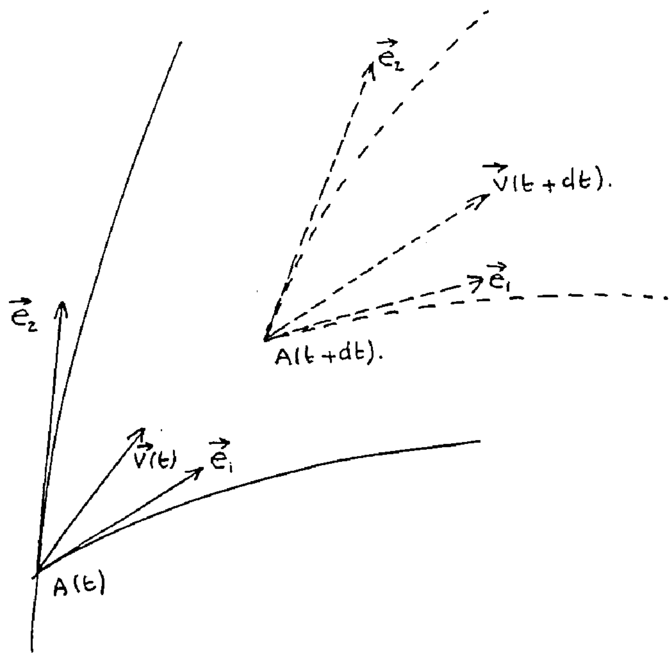
Isomorphism of the space and body coordinate systems in state t_0.

The dash-dot line represents the space coordinate system, the dashed line the body coordinate system at time t and the solid line the body coordinate system at time t_0.



Fig(4)

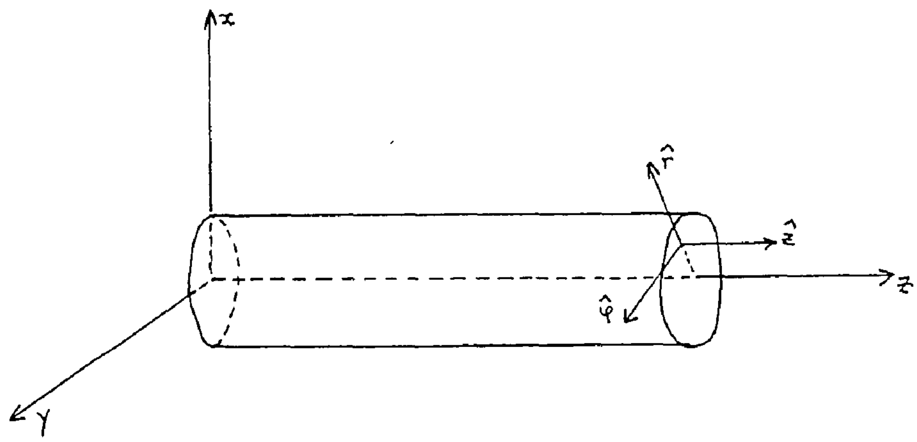
Force and normal on a body surface



Fig(5)

Acceleration vector in a body coordinate system

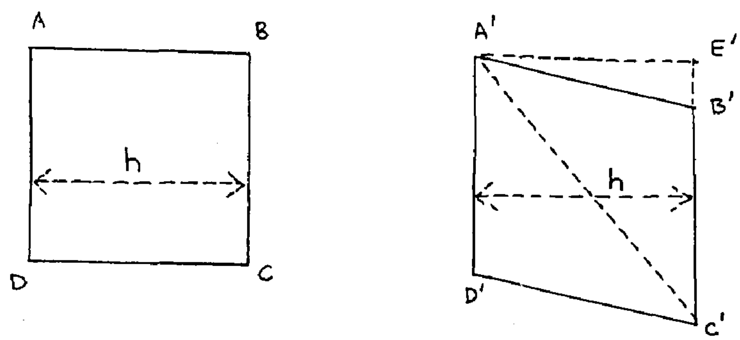
The solid line represents the body coordinate system at time t and the dashed line the body coordinate system at time t+dt.



Fig(6)

Coordinate system associated with the circular cylinder

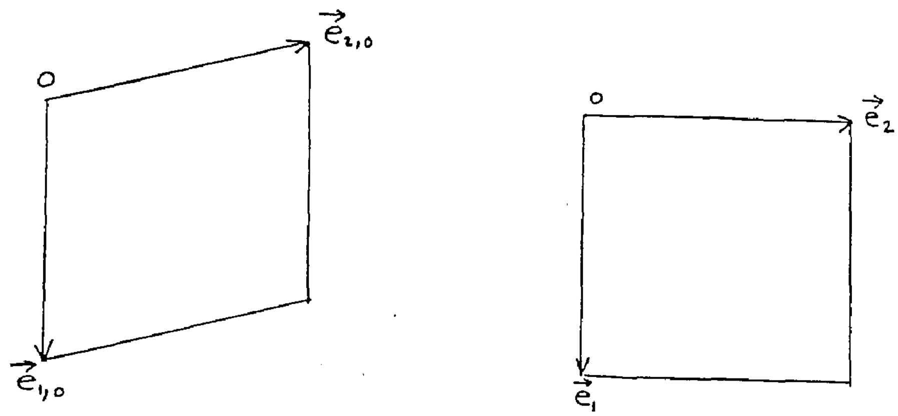
State t_o State t



Top view along r direction

Fig(7)

Deformation of an infinitesimal volume element.

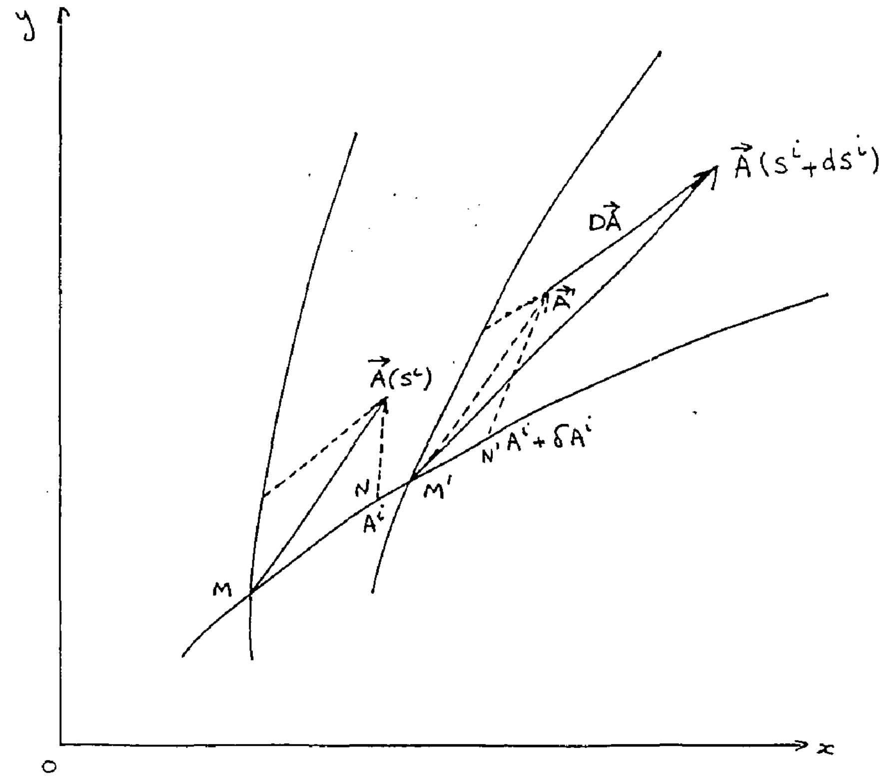


State t_o State t

Top view along r direction

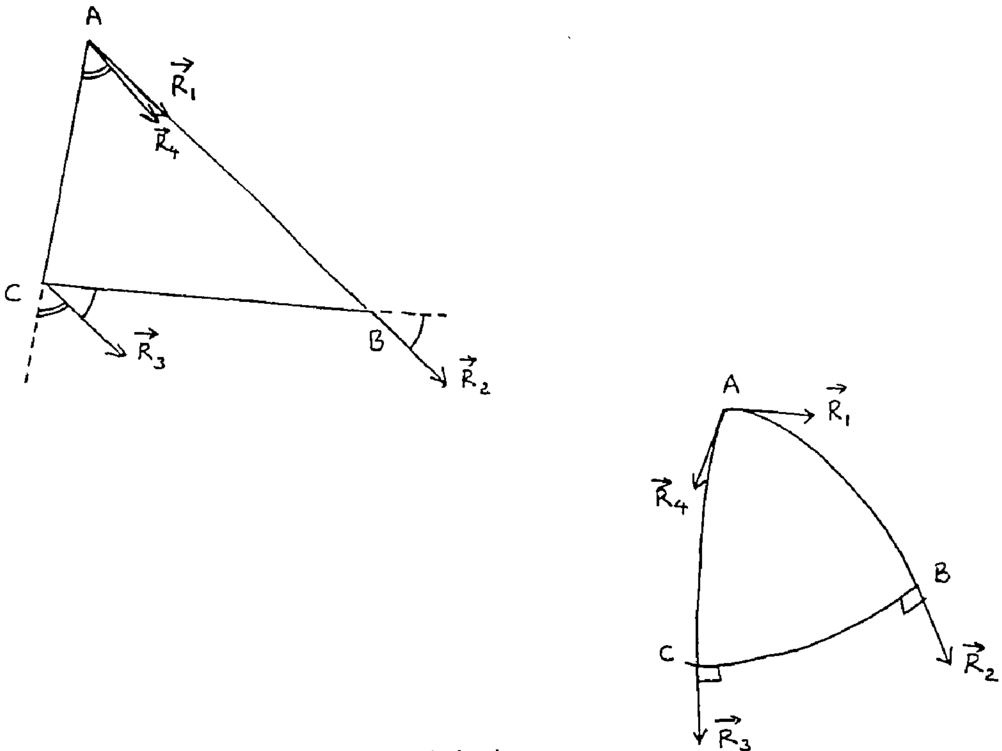
Fig(8)

Deformation of an infinitesimal volume element with an isomorphism of the body and space coordinate systems in state t.



Fig(9)

Differential of a vector in a curvilinear coordinate system.



Fig(10)

Parallel transport of a vector.

In (a) R_1 and R_4 coincide, but for the sake of clarity one has separated them.

TABLES

Table 1

Elastic constants

In units of 10^{11} dynes/cm^2

Materials	μ	λ	A'	B'	C'
Steel[13]					
Helca 37(0.4 percent C)	8.21	11.1	-70.8	-28.2	-17.9
Helca 17(0.6 percent C)	8.20	11.05	-66.8	-26.1	-6.7
Helca 138 A	8.19	10.9	-70.8	-26.5	-16.15
Rex 535 Ni Steel	8.18	10.9	-67.8	-24.0	-8.75
Helca ATF austentic	7.16	8.7	-40.0	-55.2	1.7
Aluminium alloys[13]					
2S	2.67	5.70	-40.8	-19.7	-11.4
B53S M	2.60	5.80	-27.6	-9.9	-12.5
B53S P	2.62	6.19	-30.0	-15.5	-4.65
D54 S	2.60	4.91	-32.0	-19.8	-18.9
JH 77S	2.68	5.75	-43.6	-17.7	-16.0
Magnesium tooling plate[13]	1.66	2.59	-16.8	-5.74	-3.27
Molybdenum[13]					
Sintered	11.0	15.7	-77.2	-28.3	-2.55
Resintered	12.4	17.8	-90.8	-39.8	9.70
Tungsten[13]					
Sintered	7.30	7.50	-49.6	-14.3	-10.75
Resintered	13.7	16.3	-106.8	-25.8	-21.50
Columbium[15]	3.75	14.5	30.0	-37.0	-24.0
Polysterene[14]	0.14	0.29	-1.00	-0.83	-1.06
Fused Silica[16]	3.12	1.61	-5.28	5.40	21.5
Pyrex[14]	2.75	1.35	42.0	-11.8	13.2

Table 2

Materials	Γ	Q2	Q3	Q4
Tungsten				
Sintered	1.02	-1.70	-0.49	-0.37
Resintered	1.18	-1.95	-0.47	-0.38
Steel	1.35	-2.15	-0.86	-0.54
Molybdenum				
Sintered	1.43	-1.75	-0.64	-0.06
Resintered	1.43	-1.83	-0.80	0.19
Magnesium	1.56	-2.52	-0.86	-0.49
Aluminium alloy	2.06	-3.70	-1.78	-1.03
Columbium	3.87	4.00	-2.46	-1.60
Pyrex	0.492	3.82	-1.07	1.2
Fused Silica	0.51	-0.422	0.432	1.72
Polysterene	2.09	-1.81	-1.50	-1.92

REFERENCES

1 L.I.Sedov, Introduction to the mechanics of a continious medium (Addison-Wesley Publishing Co., 1965).

2 A.S.Lodge, Elastic liquids (Academic Press, 1964).

3 A.S.Lodge, Body tensor fields in continuum mechanics (Academic Press, 1974).

4 W.Prager, Introduction to the mechanics of continua, (Ginn and Co, 1961).

5 A.D.Freed, A body tensor formalism in elastic-plastic continua with applications in uniaxial extension, (Ph.D., University of Wisconsin, 1985).

6 L.I.Sedov, A course in continium mechanics Vol I and Vol IV. (Walters-Noordhoff Publishing Co, 1971).

7 L.Landau and E.Lifchitz, Elasticity (J. Wiley and Sons N.Y. 1959).

8 A.E.Green and W.Zerna, Theoretical elasticity, second edition (Clarendon Press 1968).

9 F.D.Murnaghan, Finite deformations of an elastic solid (John Wiley and Sons, N.Y., 1951).

10 A.S.Lodge, Reol. Acta. 11 106-118 (1972).

11 M.Rainer, Israel Journal of Technology 4 187-192 (1966).

12 A.E.Green and J.E.Atkins, Large elastic deformations, second edition (Clarendon Press, 1970).

13 R.T.Smith, R.Stern and R.W.B.Stephens, J.Acoust. Soc. Am. 40 1002 (1966).

14 R.T.Smith, Ultrasonics, July-September 1963 p 135

15 L.J.Graham, H.Nadler and R.Chang, J.Appl.Phys. 39 3325 (1968).

16 W.T.Yost and M.A.Breazeale, J.Appl.Phys 44 1909 (1973).

17 V.K.Raju and P.J.Reddy. J.Phys. D Appl. Phys. 14 65 (1981).

18 R.A.Toupin and B.Bernstein J.Acoust.Soc.Am. 33 216 (1961).

19 Z.A.Goldberg Soviet Physics-Acoustics 6 306 (1961).

20 A.V.Skachenko and A.N.Sporykhin P.P.M. 41,No 6, 1145-1146 (1977).

21 L.Landau and E.Lifshitz, The classical theory of fields (Addison-Wesley Publishing Co. 1951).

22 A.Lichnerowicz Elements of tensor calculus (John Wiley and Sons N.Y., 1962).

23 I.S.Sokolnikoff Tensor analysis. (J.Wiley and Sons, 1951).

24 G.Wertheim, Annales de Chimie et de Physique,third series 50, 195-321 and 385-431 (1857).

25 J.H.Poynting, Proc. Roy. Soc. A82 546 (1909).

26 J.H.Poynting, Proc. Roy. Soc. A86 543 (1912).

27 J.Bell, Encyclopedia of physics. Vol VIa/1 C.Truesdell,. editor. (Springer-Verlag, N.Y. 1973).

28 A.Foux, Proceedings, International symposium on second-oder effects in elasticity,plasticity and fluid mechanics. Haifa, Israel. 228. (Pergamon Press, N.Y., 1964).

29 J.G.Oldyrod, Proc. Roy. Soc. A200 523 (1950).

30 J.G.Oldyrod, Proc. Roy. Soc. A245 278 (1958).